Option Pricing in Lévy Process Model with Stochastic Volatility using the Finite Difference Method

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Abstract

This paper considers option pricing when an underlying security’s price follows Lévy process and the volatility is stochastic. Then in the general case closed-form solution is impossible to find. This paper applies a multi-dimensional finite difference method, ADI to solve a suitable partial integro-differential equation and find European and American option prices.

Keywords: Option pricing, finite difference methods, Lévy process, stochastic volatility

JEL classification: C63, D53, G12
1 Introduction

Black and Scholes (1973) find a closed-form solution of European option price. Practitioners actively used option pricing model has gotten a lot of critique. The model’s assumptions about stock price volatility, deterministic interest and stock prices which follow geometric Brownian motion have been seen unrealistic. The most famous example of problem with Black Scholes model is volatility smile i.e. the implied volatility of different strike prices is not same but volatility is decreasing for small values and increasing for big values of strike price.

It is very easy to notice by empirical study that the assumption about stock prices which adhere to geometric Brownian motion is too restrictive. In practice, quite rare, abrupt upward and downward jumps are typical for stock prices. Empirical data often exhibit more small values than normal law and stock price distributions have often heavy tails which kind of behavior Brownian motion can not describe. To get the correct price of option it is necessary to get realistic description for the behavior of option price.

One possible extension of Black Scholes model is to make more general assumption about the stock price process. In the academic literature often applied process is Lévy process which special cases e.g. Brownian motion and Poisson process are. Jump-diffusion process which merges a continuous Brownian motion and some discontinuous jump-process is also a Lévy process (See more in Applebaum (2004)).

Merton (1976) develops an option pricing model in the case in which the value of underlying stock is the mixture of continuous Ito diffusion process and discontinuous Poisson process. Other examples of special cases of Lévy process which has been used in the option pricing literature is double exponential jump-diffusion (Kou(2004)) and variance gamma process (Carr et all. (2002)).

Black and Scholes (1971) use arbitrage-pricing to get the price of European option using arbitrage-pricing. Cox et al. (1985a) get the corresponding solution in a continuous-time general equilibrium framework and find a partial differential equation which gives the equilibrium price of the option in terms of the underlying variables in the economy.
Solving numerically Cox et al. (1985a) type parabolic partial differential equation it is possible to get the precise value of option. The numerical solution of PDE has turned out to be very useful in the option pricing because of its flexibility about the assumption of the model’s parameters. The PDE is possible to set up different way depending on the assumption of a model but it is still solvable numerically.

Contrary to cases with Black Scholes assumptions in the case of Lévy distributed returns there are generally no closed form solution of option price because the probability density of a Lévy process is not known. Thus, only way to get a solution in the case of Lévy process is to apply numerical methods.

There are two main approach to find the numerical solution of PDE: Monte Carlo simulation and finite difference method (FDM)/ finite element methods. When the dimension of the option pricing problem increases Monte Carlo simulation becomes faster or only applicable approach. But when there are 4-5 or less dimensions finite difference method is at least as computationally efficient than Monte Carlo simulation.

During the decades finite difference method has been used for solving the partial differential equations in the physical problems (for an introduction see e.g. Richtmeyer and Morton, 1967). Schwartz (1977) and Brennan and Schwartz (1978) as the first suggest finite difference method for option pricing. Randall and Tavella (2000) describe diversified ways to apply finite difference method to the instrument pricing problems.

During the last couple of years the studies of option pricing models in which stock returns follow jump-diffusion and volatility is deterministic has been active. But there are also a few older paper.

Already Merton (1976) argues that the assumption about securities which follow Brownian motion is restricted. As said, he develops an option pricing model in the case in which the value of underlying stock is the mixture of continuous Ito diffusion process and jump process using arbitrage-pricing. Merton model is solvable by arbitrage pricing because he assumes that there are no correlation between the jumps and the market portfolio. If there are correlation between Ito diffusion and jump uncertainties, the financial markets are not complete i.e. arbitrage-

In the case of Levy process it is possible to form so called integro-differential equation (PIDE) and solve it using finite difference method. Tavella and Randall (2000) propose straightforward method to solve European option price. They use an implicit time discretization and solve the problem with full matrix.

A few articles propose computationally more efficient methods in the case Levy process. Andersen and Andreasen (2000) speed up the computation combining an alternating direction implicit (ADI) finite difference method and fast Fourier transformation (FFT). Their method involves two fractional step: during the first half-step, the spatial differential operator is treated using implicit FDM and the integral operator is treated using explicit FDM and the during the second half step another way round.

An European option can be exercised only at the expiry date while an American option allows holder to choose the time of exercise during the life of the option. Toivanen (2008) solves American option price under Kou’s (2002) jump-diffusion process. He uses central finite difference scheme for spatial discretization and Rannacher stepping scheme and a nonuniform grid for time discretization and derives recursion formulas for solving the convolution integrals. Almendral and Oosterlee (2005) solves European American option prices under the variance gamma process of Carr et al. 2002. They use uniform grids and an explicit backward differentiation formula. Cont and Voltchkova (2005) examine the solution of option price when stock follows jump diffusion and exponential Levy process. They propose an finite difference method which is based on splitting the operator into a local and a nonlocal part. They treat the local term using an implicit step and nonlocal term using an explicit step. Zhang (1997) finds the price of American option in the case of Merton’s jump-diffusion model using a finite difference method. They treat integral term of PIDE explicitly in time and the differential terms of PIDE implicitly. That method is first-order accurate in time.

Assuming stochastic volatility is another way together with Lévy process fol-
ollowing stock returns to improve Black Scholes model. Hull and White (1987) used Monte Carlo simulation to solve option prices with the stochastic volatility of an underlying security. They form a PDE which is possible to solve also using finite difference method. Heston (1993) finds a closed solution in the case of Ornstein-Uhlenbeck process. In the academic literature, it is also assumed that interest rate is not constant and follows some stochastic process (e.g. Amin ja Ng (1993)).

In this paper, I propose a FFT-ADI method for case where the underlying stock price adheres to Levy process and its volatility is stochastic. I combine explicit-implicit and FTT-ADI methods to get fast method which stability properties are smooth and precise. Using the method I solve European and American option prices.

The rest of the paper is structured in the following way. Chapter 2 gives some set-ups and defines the option model of this paper. Chapters 3 considers a applicable part of FDM’s theory, shape a proper PIDE and considers how to solve it. Chapter 4 presents the results of computation in case of European and American option and consider the property of the method. Finally, chapter 5 is for conclusion and for proposing some ideas of further research.

2 Option pricing model

2.1 Securities

Black and Scholes (1973) model gives closed-form solution for the price of European option. They suppose that the price of underlying stock $S$ follows dynamics

$$dS_t = S_t(\alpha dt + \sigma dz_t)$$

(2.1)

where $\alpha$ and $\sigma^2$ are the instantaneous percentage change price per unit time of the stock and the instantaneous variance per unit time of the stock. $z_t$ is a standard Brownian motion on a probability space $(\Omega, F, P)$. Because the interest rate $r$ is constant, all uncertainty in Black Scholes model is given by realizations of Brownian motion process.
A partial differential equation corresponding to Black Scholes model’s assumption has been found by Cox et al. (1985a). The log-price of an option \( f(t, S) \) must satisfy

\[
\frac{\partial f(t, S)}{\partial t} + rS \frac{\partial f(t, S)}{\partial S} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f(t, S)}{\partial S^2} = rf(t, S)
\]  

(2.2)

with suitable boundary conditions.

### 2.1.1 The Stochastic Volatility Problem

Hull and White (1987) extends Black Scholes case and solve option when asset price follows the diffusion

\[
dS_t = \alpha S dt + \sqrt{v_t} S dz^1_t
\]

(2.3)

and where the volatility is

\[
d\sqrt{v_t} = -\beta \sqrt{v_t} dt + \delta dz^2_t
\]

(2.4)

where Brownian motions \( z^1_t \) and \( z^1_t \) have correlation \( \rho \) and \( \beta, \delta \) and are constants.

Then the option value satisfy the PDE

\[
\frac{\partial f}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial S^2} + rS \frac{\partial f}{\partial S} + \rho \sigma \delta S \frac{\partial^2 f}{\partial S \partial v} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial v^2} + \alpha \sigma^2 \frac{\partial f}{\partial v} = rf.
\]

This PDE is also easy solve by the finite difference methods.

### 2.1.2 Lêvy process

This subchapter considers an \( \mathcal{F}_t \)-adapted process \( \{X(t)\}_{t \geq 0} \) with filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) such that \( e^X \) is a martingale with respect to the canonical filtration \( \mathbb{F}_t^0 = \sigma(X_s, s \leq t), t \geq 0 \).

**Definition 1.** A cadlag stochastic process \( \{X_t : t \geq 0\} \) on \( \mathbb{R} \) such that \( X_0 = 0 \) a.s. is called a Lêvy process if it has independent and stationary increments and is stochastically continuous.
The martingale condition implies
\[ \int_{|z|>1} e^{z} \upsilon(dz) < \infty \quad (2.6) \]

**Assumption 1.** The Lévy measure \( \upsilon \) of \( X \) admits *semiheavytails*: Let \( \upsilon \) denote the marginal Lévy measure. We assume that there are constants \( \beta^- > 0 \) and \( \beta^+ > 0 \) such that
\[ \int_{1}^{\infty} e^{\beta^+ z} \upsilon(dz) < \infty \quad \text{and} \quad \int_{-\infty}^{-1} e^{-\beta^- z} \upsilon(dz) < \infty \quad (2.7) \]

Now, the risk-neutral dynamics of stock price \( S \)
\[ S_t = S_0 e^{rt + X_t} \],
where \( X_t \) is Lévy process.

**Theorem 1.** (Lévy-Khintchine representation)

Let \( \{X(t)\}_{t \geq 0} \) be a Lévy process with Lévy measure \( \upsilon \) satisfying \( \upsilon(\{0\}) = 0 \) and \( \int \upsilon(dx) < \infty \). Then \( \int_{\mathbb{R}} \min(1, z^2) \upsilon(dz) < \infty \) and for the characteristic function \( \Phi_t \) of \( X \) at time \( t \geq 0 \) holds
\[ E[e^{isX_t}] = e^{t\Phi(s)}, \quad s \in \mathbb{R}^d \quad (2.8) \]
\[ \Phi(s) = ias - \frac{1}{2}s\sigma + \int_{|z|<M} (e^{isz} - 1 - isz) \upsilon(dz) + \int_{|z|\geq M} (e^{isz} - 1) \upsilon(dz) \quad (2.9) \]
with \( a \in \mathbb{R}, \sigma, M \in \mathbb{R}^+ \).

Definition 2 (follows Øksendal and Sulem (2007)) gives an operator in the case of Lévy process which corresponds to a differential operator.

**Definition 2.** Let \( X \) be a Lévy process. Then the infinitesimal generator of \( X \) is defined as an integro-differential operator defined by the expression
\[ \mathcal{L} f(x) = \lim_{t \to 0^+} \frac{E^x[f(X^{(x)}(t))] - f(x)}{t} \quad (2.10) \]
(if the limit exists)
where \( X^{(x)}(0) = x \).
If function $f \in C^2_0(\mathbb{R})$, then $\mathcal{L}f(x)$ exists and is given by

$$\mathcal{L}f(x) = a \frac{\partial f}{\partial x}(x) + \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} + \int_{\mathbb{R}} \nu(dy)[f(f + y) - f(x) - y1_{\{|y|<1\}}\frac{\partial f}{\partial x}(x)], \quad t \geq 0$$

(2.11)

where the last term is convolution integral of $x$ and $y$.

The variance gamma process is a special case of Lévy process. Carr et al. (2002) apply it to the option pricing. The variance gamma process can be interpreted as a Brownian motion with drift, where time is changed by a gamma process $\Gamma(\frac{t}{\nu})$. The density of the gamma process at time $t$ is

$$f(g) = \frac{g^{t/\nu} - e^{-g}}{\nu^{t/\nu} \Gamma(\frac{t}{\nu})}$$

(2.12)

and its another formulation by Carr et al. (2002) is

$$k(y) = C e^{-G|x|}, \quad if \quad y < 0$$

$$k(y) = C e^{-M|x|}, \quad if \quad y > 0$$

(2.13)

for constants $C > 0$, $G \geq 0$, $M \geq 0$ and $\alpha < 2$. A special case with parameter value $\alpha = -1$ is another regularly used distribution in literature the log-double-exponential density (especially Kou (2004)).

### 2.1.3 European option

The payoff of an European call option holder at time $T$ is

$$F(t, S) = (S(T) - K)^+ = max\{S(T) - K, 0\}$$

(2.14)

where $K$ is the exercise price. In the case of geometric Lévy process distributed stock price, the option price can be shown using equation (2.11). Let maturity time be $T$, strike price $K$ and $f(s, t)$ the log-price of option. The solution of following Cauchy problem gives log-price:

$$\frac{\partial u}{\partial t} - \mathcal{L}u = 0, \quad [0, T] \times \mathbb{R} \rightarrow \mathbb{R},$$

(2.15)
\[ u(0, S) = \phi_0 \]

where \( \phi_0 \) is known.

### 2.1.4 American option

American option allows holder to choose the time of exercise. The early exercise possibility leads to a constraint for the value of American options. The value at time \( t \) of an option that can be exercised at time \( \tau \) is given by

\[
g(r, S, t) = \sup_{\tau \leq T} E\left[ e^{-\int_t^T r(s) ds} f(S(\tau)) | \mathcal{F}_t \right]
\]  

(2.16)

The price of American call option lead to inequalities which have a form of linear complementary problem (See e.g. (Tavella and Randall (2000)).

\[
g(t, S) \geq f(T, S),
\]

\[
\mathcal{L}g(\tau, S, t) \leq rf(t, S)
\]

\[
[\mathcal{L}g(\tau, S, t) - rf(\tau, S, t)] [g(\tau, S, t) - f(T, S)] = 0
\]

(2.17)

\[
g(\tau, T, S) = f(T, S)
\]

For computational purpose, it is useful to rewrite inequations in (2.17). Then the linear complementary problem is:

\[
\left[ \frac{\partial g(\tau, S, t)}{\partial t} - \mathcal{L}g(\tau, S, t) \right] \geq 0,
\]

\[
g(t, S) \geq f(T, S),
\]

(2.18)

\[
\left[ \frac{\partial g(\tau, S, t)}{\partial t} - \mathcal{L}g(\tau, S, t) \right] [g(\tau, S, t) - f(t, S)] = 0
\]

### 2.2 Partial Integro-Differential Equations

The partial differential equation related to Black Scholes model and Heston model are presented in (2.2) and in (3.11) corresponding equation in the case Lévy process is partial integro-differential equation (see e.g. Oksendal and Sulem (2007)).
**Theorem 2.** Let $X$ be a Lévy process with state space $\mathbb{R}$. Assume that the function $f(t, S)$ satisfies

$$f(t, S) \in C^{1,2}((0, T) \times R^+) \cap C^0([0, T] \times R^+) \quad (2.19)$$

Then $f(t, S)$ satisfies the PIDE

$$\frac{\partial f}{\partial t} + r S \frac{\partial f}{\partial S} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial S^2} - r f + \int_{-\infty}^{+\infty} (f(t, e^z S) - f(t, S) - S(e^z - 1) \frac{\partial f}{\partial S}) \nu(dz) = 0 \quad (2.20)$$

in $(0, T) \times R^+$ and the terminal condition is given by

$$f(T, S) = g(S) \quad \forall S \in \mathbb{R}^+ \quad (2.21)$$

Also the partition on the following theorem is useful for the computation.

**Theorem 3.** (Hilber et al.(2009)) Let $X$ be a Lévy process with state space $\mathbb{R}$ and $\nu$ Lévy measure satisfying assumption 1, $\beta^+ > 1$ and $\beta^- > 0$. Let

$$u(\tau, x) = e^{r\tau} f(T - \tau, e^{x+(\gamma - r)\tau}) \quad (2.22)$$

where

$$\gamma = \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^z - 1 - z) \nu(dz) \quad (2.23)$$

Then $u$ satisfies the PIDE

$$\frac{\partial u}{\partial t}(t, S) + A_1(u) + A_2(u) = 0 \quad (2.24)$$

in $(0, T) \times \mathbb{R}$ with initial condition $u(0, x) \equiv u_0$. In (2.24)

$$A_1(u) = -\frac{\sigma}{2} \frac{\partial^2 \phi}{\partial x_i x_j} \quad (2.25)$$

and

$$A_2(u) = -\int_{\mathbb{R}} (\phi(x + z) - \phi(x) - z \frac{\partial \phi}{\partial x}(x)) \nu(dz) \quad (2.26)$$

are defined for $\phi \in C^2_0(\mathbb{R})$ and the initial condition is

$$u_0 = g(e^x). \quad (2.27)$$
2.3 Fast Fourier Transformation and convolution

Fourier transform has been used often in option pricing literature

\[ F[g(x)](z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixz}g(x)dx, \quad z \in \mathbb{R} \]  \hspace{1cm} (2.28)

\[ g(T, \omega) = \int_{-\infty}^{\infty} \kappa(\omega - v; t; T)\psi(v)dv \]  \hspace{1cm} (2.29)

which is convolution of two density functions \( \xi \) and \( \psi \). The convolution can be approximated by

\[ c(T, \tilde{\omega}) \approx \sum_{j=-N/2+1}^{N/2} \kappa_{i-j}\psi_j \]  \hspace{1cm} (2.30)

3 The Method

3.1 A Three-Dimensional Alternating-direction Implicit-Explicit Scheme

Alternating-direction implicit (ADI) methods are computationally simple way to solve multi-dimensional elliptic partial differential equations. The convergence and stability properties of ADI are well-known. In the many cases the method is stable up to 6 dimensions. (see e.g. Craig and Sneyd, 1988)

Two-dimensional problems can be reduced to the solution of two tridiagonal matrix equations per time-step. ADI schemes are available for multi-dimensional parabolic equations in the absence of the cross derivative terms which means that the instantaneous correlation between the state variables is zero. I solve initial value problems of the form \((2.15)\) which is in the case of this paper is three-dimensional partial differential equation. The first dimension is the description of the standard diffusion, the second dimension is the diffusion process of volatility and the third dimension is for the integral term in PIDE.

Let \( u(x) \) be a real-valued function defined on a grid in d-dimensional space is the form

\[ \mathbb{I} = \mathbb{I}_1 \times \mathbb{I}_2 \times \ldots \times \mathbb{I}_d \]  \hspace{1cm} (3.1)
where each $I_h$ is given by

$$I_h = \{\ldots, -2n_h, -n_h, 0, n_h, 2n_h, \ldots\} \quad (3.2)$$

Then finite difference operators for first-order derivatives are

$$\delta_x u_{i,j}(x, t) = \frac{u_{i,j}(x + \Delta x, t) - u_{i,j}}{2\Delta h}, \quad (3.3)$$

with accuracy $O(\Delta x^2)$. The operators for the second-order derivatives are

$$\delta^2_x u_{i,j}(x) \approx \frac{u(h_{i,j}) - 2u(h_{i,j}) + u(h_{i,j})}{\Delta h^2}. \quad (3.4)$$

with accuracy $O(\Delta x^2)$ and for the cross-derivatives of $x$ and $y$

$$\delta^2_{xy} u_{i,j}(x) \approx \frac{u(x_{i,j}) - 2u(x_{i,j}) + u(x_{i,j})}{\Delta x \Delta y}. \quad (3.5)$$

with accuracy $O(\Delta x^2) + O(\Delta y^2)$.

The numerical scheme of an ADI is

$$Au^{n+1} = (A + B)u^n \quad (3.6)$$

where

$$A = \prod_{i=1}^{P} (1 - \theta r q_{ii} \delta^2_{xi}) \quad (3.7)$$

and

$$B = r \sum_{i=1}^{P} q_{ii} \delta^2_{xi} + \frac{1}{2} r \sum_{i=2}^{N} \sum_{j=1}^{i-1} q_{ij} \delta_{xixj}, \quad (3.8)$$

where $P$ is the number of dimensions. The idea of ADI-method is to introduce intermediate time layers in the solution process where the side of time steps are $\Delta t P/2$. The effect is to solve the multi-dimensional problem as a set of one-dimensional problems.

Let consider the case of three space dimensions $x, y, z$ and (one time dimension). Then an ADI approximation is

$$\frac{\phi_{i,j}^{k+1/3} - \phi_{i,j}^k}{\delta t/3} = \frac{\phi_{i+1,j}^{k+1/3} - 2\phi_{i,j}^{k+1/3} + \phi_{i-1,j}^{k+1/3}}{(\delta x)^2} \quad (3.9)$$
\[
\phi^{k+2/3}_{ij} - 2\phi^{k+2/3}_{ij} + \phi^{k+2/3}_{ij} (\delta y)^2 + \phi^{k}_{ij} - 2\phi^{k}_{ij} + \phi^{k}_{ij} (\delta y)^2
\]  
(3.9)

ADI method is implicit in x and explicit in y and in z. I follow Anderson and Andreasen (2000) in the application of the ADI method to Lévy process. In addition, I use the third dimension to model stochastic volatility.

### 3.2 A Partial Integro-Differential Equation

The use of finite difference method to solve option prices in the Lévy process case based on the solution of PIDE equation (3.11). To solve the problem of this paper it is necessary to merge the PDE (3.11) and PIDE (3.11) Then we get PIDE for \( f(t, S) \) (Again, it is assumed that the function \( f(t, S) \) satisfies

\[
\begin{aligned}
\frac{\partial f}{\partial t} + & \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial S^2} + r S \frac{\partial f}{\partial S} \\
& + \rho \sigma^2 \delta S \frac{\partial^2 f}{\partial S \partial v} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial v^2} + \alpha \sigma^2 \frac{\partial f}{\partial v} + r f \\
& + \int_{-\infty}^{+\infty} (f(t, e^S z) - f(t, S) - S(e^z - 1) \frac{\partial f}{\partial S}) v(dz) = 0
\end{aligned}
\]  
(3.11)

in \((0, T) \times \mathbb{R}^+\). The terminal condition is given by

\[
f(T, S) = g(S) \quad \forall S \in \mathbb{R}^+
\]

The parabolicity of PDE (3.11) implies that the symmetric matrix \( Q = (q_{ij}) \) is positive definite and problem is well-defined.

#### 3.2.1 Approximating integrals

The most difficult task when a numerical scheme is used in Lévy case is to solve the integral in (3.11). At the beginning of the solution the integral part (3.11) has to truncate to a bounded interval \([-A, A], A > 0\). Cont and Voltchkova (2002) study precisely the behavior of the truncation error. It can be shown that the truncation error decays exponentially with respect A. In the computational examples
of chapter 4, the lower and upper bounds of the interval $A_l, A_h$ have been chosen so that the truncation error insignificant small.

Before numerical solution, it is useful to decompose equation (3.11) using theorem 3.

$$\frac{\partial f}{\partial t}(t, S) + D_1 + D_2 + J = 0 \quad (3.12)$$

where

$$D_1 = \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial S^2} + \left(-\frac{\sigma^2}{2} + r + \alpha\right) \frac{\partial f}{\partial S} - \lambda f$$

$$D_2 = \rho \sigma^3 \delta S \frac{\partial^2 f}{\partial S \partial v} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 f}{\partial v^2} + \alpha \sigma^2 \frac{\partial f}{\partial v} - rf,$$

and

$$J u = \int_{-A}^{A} v(dy) u(n, x_i + y) \quad (3.13)$$

and $\alpha = \int_{-A}^{A} v(dy)(\delta x, x + y)$. The trapezoidal quadrature rule with the same uniform steps can be used for the approximation of the integral term (3.13):

$$J u \approx J_{\Delta} u = \sum_{j=K_l}^{K_h} v_j u_{i+j} \quad (3.14)$$

Approximating the integral part we obtain

$$J u = \int_{-A}^{A} v(dy) u(n, x_i + y) \approx \sum_{j=-A_l}^{A_h} v_j u_{i+j} \quad (3.15)$$

$$\gamma \approx \tilde{\gamma} = \sum_{j=K_l}^{K_r} (e^{y_j} - 1)v_j \quad (3.16)$$

where $v_j = \int_0^{(j+1/2)/\Delta x} v(dy)$.

I introduce an uniform grid on $[0, T] \times [-R, R]$ with time step $\Delta t = T/M$ and mesh width $\Delta x = 2R/N$ for $N, L \in \mathbb{N}$. To approximate the integral terms I use the trapezoidal quadrature rule.
3.2.2 Discretization

The space derivatives are discretized using central finite difference operators (3.3), (3.4) and (3.5). Then the differences which correspond the cross-derivatives are:

\[
u(x_{i+1}, x_{j+1}) \approx u + \Delta x \frac{\partial u}{\partial x} + \Delta y \frac{\partial u}{\partial y} + \frac{1}{2} (\Delta x^2 \frac{\partial^2 u}{\partial x^2} + 2 \Delta y \Delta x \frac{\partial u}{\partial x \partial y} + \Delta y^2 \frac{\partial^2 u}{\partial y^2})
\]

(3.17)

I apply Anderson and Andreasen’s (2000) 2-dimensional scheme to 3-dimensional case and use fractional stepping:

\[
0 = \frac{1}{\Delta t} (f(t + \Delta t) - f(t)) + D_1[\theta_1 f(t) - (1 - \theta_1) f(t + \Delta t)] + D_2[\theta_2 f(t) - (1 - \theta_2) f(t + \Delta t)] + \lambda(-1 + \varphi^*)[\theta_3 f(t) + (1 - \theta_3) f(t + \Delta t)]
\]

(3.18)

where \(\theta_i \in [0, 1]\) \(\forall i = 1, 2, 3\) are constants.

Rearranging 3.19 gives

\[
[\Delta t^{-1} - \theta_1 D_1 - \theta_2 D_2 - \theta_3 \lambda(-1 + \varphi)] f(t) = [\Delta t^{-1} + (1 - \theta_1) D_1 + (1 - \theta_2) D_2 + (1 - \theta_3) \lambda(-1 + \varphi)] f(t + \Delta t)
\]

(3.19)

The different value of \(\theta_i\) are theoretically possible. If we put \(\theta_i = \frac{1}{2} \forall i\) we obtain Crank-Nicolson scheme. It is practically non-feasible, because it gives computationally cost full matrices. Hull and White (1990) find that explicit method is faster than implicit method. On the other hand, explicit methods generally suffer from instability problems as well as poor convergence in the time-domain. Implicit methods exhibit better precision, convergence and stability properties. As generally is known, explicit scheme, \(\theta_i = 0 \forall i\) is unstable and its convergence in the time domain is only \(O(\Delta t)\) (see e.g Andersen and Andreasen (2000)). Each time-step in the grid is split into three third-steps. For the first third-step is set \(\theta_1 = 1\) and \(\theta_2 = \theta_3 = 0\)

\[
\left[\frac{1}{\Delta t/3} + D_1\right] (f(t + \Delta t/3) = \left[\frac{1}{\Delta t/3} + D_2 + \lambda(-1 + \varphi)\right] (f(t + \Delta t/3)
\]

For the second third-step is set \(\theta_2 = 1\) and \(\theta_1 = \theta_3 = 0\)

\[
\left[\frac{1}{\Delta t/3} + D_2\right] (f(t + \Delta t/3) = \left[\frac{1}{\Delta t/3} + D_1 + \lambda(-1 + \varphi)\right] (f(t + \Delta t/3)
\]

(14)
For the third third-step is set $\theta_3 = 1$ and $\theta_1 = \theta_2 = 0$

$$\left[\frac{1}{\Delta t/3} + \lambda(-1 + \varphi)\right](f(t + \Delta t/3) = \left[\frac{1}{\Delta t/3}D_1 + D_2\right](f(t + \Delta t/3)$$

### 4 Numerical Results

In this section is given an example of numerical solution of the method that has been proposed in the previous chapter. Using the method, the price of European and American call options are solved corresponding to the value of economic parameters in the model. In this example has been considered case where the value of economic parameters are $r = 0.05$, $q = 0.02$, $\sigma = 0.15$, $\rho = 0.5$, $\lambda = 0.1$, $\gamma = 0.4$, $S(0) = 100$, $K = 100$. I also apply a special case of Lévy process, CGMY process (presented in subsection (2.1.2)) to the option pricing problem of this paper. I have set the values of parameters following Cont and Voltchkova (2002): $C = 6.25$, $G = 14.4$ and $M = 60.2$. Some other value of the parameters or another formulation of process would be surely possible.

In table (1) is shown European option prices solved by the method of this paper.

<table>
<thead>
<tr>
<th>x-steps</th>
<th>T=0.01</th>
<th>T=1</th>
<th>T=10</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>0.612</td>
<td>8.542</td>
<td>20.431</td>
</tr>
<tr>
<td>64</td>
<td>0.6431</td>
<td>8.987</td>
<td>20.557</td>
</tr>
<tr>
<td>128</td>
<td>0.6554</td>
<td>9.112</td>
<td>20.611</td>
</tr>
<tr>
<td>256</td>
<td>0.6578</td>
<td>9.223</td>
<td>20.624</td>
</tr>
<tr>
<td>512</td>
<td>0.6581</td>
<td>9.231</td>
<td>20.627</td>
</tr>
<tr>
<td>1024</td>
<td>0.6581</td>
<td>9.234</td>
<td>20.629</td>
</tr>
</tbody>
</table>

Using the PIDE in chapter 3 the value of American option can be given using the notation of chapter 3:

$$\frac{\partial u}{\partial t} + A_1(u) + A_2(u) \leq 0, \quad \text{in} \quad [0, T] \times \mathbb{R},$$
\[ u(\tau, \cdot) \geq \tilde{g}_\tau, \text{ in } [0, T] \times \mathbb{R}, \]
\[ (u - \tilde{g}_\tau) \left( \frac{\partial u}{\partial \tau} + A_1(u) + A_2(u) \right) = 0 \quad (4.1) \]
\[ \tilde{g}(x) = e^{r\tau} g(e^{x+(\gamma-r)\tau}), \quad x \in \mathbb{R}. \quad (4.2) \]

With a reasonable amount of the effort in implementation American option prices can be solved using the previous method and these equations. In table (2) is shown American option prices solved by ADI method. I performed all numerical experiments on a 3.0-GHz Intel Core PC. The methods have been implemented by MATLAB 7.7 software.

| Table 2: American option prices using the 3-dimensional method |
|----------------------|---------------|-------------|
| 32                    | 0.7437        | 9.1212      | 22.448 |
| 64                    | 0.7671        | 9.2547      | 20.732 |
| 128                   | 0.7745        | 9.3199      | 20.781 |
| 256                   | 0.7782        | 9.3241      | 20.788 |
| 512                   | 0.7810        | 9.3255      | 20.791 |
| 1024                  | 0.7811        | 9.3256      | 20.793 |

4.1 The Properties of the Method

**Proposition 1.**

The the method of chapter 3 is unconditionally stable in the von Neumann sense.

For the case of deterministic parameters, the numerical solution of the scheme is locally accurate to order \( O(\Delta t^2 + \Delta x^2 + \Delta y^2) \). The scheme is unconditionally stable in the von Neumann sense. For the case of deterministic parameters, the numerical solution of the scheme is locally accurate to order \( O(\Delta t^2 + \Delta x^2 + \Delta y^2) \).

The proof of proposition it is easy to attain generalizing the proof of proposition 3 in Andersen and Andreasen (2000). The computational burden is \( O(\Delta t^2 + \Delta x^2 + \Delta y^2) \). There is some truncation error that tends to zero as discretization steps tend to zero. If that holds the solution of the difference equation converges to the solution of the differential equation.
Well-known Lax Equivalence Theorem says that stability is only requirement for convergence if a linear initial value problem is properly posed and FD scheme is consistent. (see e.g. Tavella and Randall (2000)) Thus, because of linearity, consistency and stability, my method also converges.

5 Conclusion

In this paper, I have applied 3-dimensional ADI method to solve the price of European and American options in a case where option price depends on Levy distributed stock returns and stochastic volatility. The method is stable and locally accurate.
References


