Monte Carlo Computation of Optimal Portfolio Choice with Habit Formation

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Problem: How rational investor allocates her wealth between consumption and existing securities?

Merton (1971) finds closed-form solutions in some restricted cases using Itô’s Lemma

Two extensions to original Merton’s problem:
- (1) more realistic assumptions about the preferences of consumer
- (2) more realistic assumptions about the behavior of financial assets
Merton (1971)

- The optimization problem of an investor with time-separable utilities

\[
\max E_0 \left[ \int_0^T e^{-\rho t} u(c(t), t) \, dt \bigg| \mathcal{F}_0 \right],
\]  

she invests to non-risky asset (deterministic interest rate) \( r_t \)
and to risky assets (stocks) which are generated by:

\[
\frac{dS_{it}}{S_{it}} = \alpha_i(S, t) \, dt + \sigma_i(S, t) \, dz_i
\]

- we can work with the 2-asset case without loss of generality;
\( \pi \) is proportion of investment in stocks

- wealth process (budget constraint): 

\[
dw_t = \pi_t w_t \left( \alpha_t + \sigma_t dz \right) + (1 - \pi_t) w_t r_t \, dt - c_t \, dt
\]

\[\iff\]

\[
dw_t = (\pi_t \mu_t + (w_t - \pi_t) r_t - c_t) \, dt + \pi_t \sigma_t dz_t
\]
The are many empirical results which indicates that utility in period t depends on not just consumption in same period but also the level of consumption in the previous periods (habit formation).

in the case of habit formation, utility function is

\[ U(h, c) = E \left[ \int_0^T e^{-\rho t} u(t, c(t) - h(t; c)) dt \big| \tilde{F}_0 \right], \quad (4) \]

where

\[ h(t) = h_0 e^{-\int_0^t a(v) dv} + \int_0^t b(s) e^{-\int_s^t a(v) dv} c(s) ds. \quad (5) \]

intuition: An individual who consumes a lot in period (t-1) will get used to that high level of consumption i.e. has high standard of living, and will more strongly desire consumption in period t
e.g. Constantinides (1990) solves the equity premium puzzle applying habit formation. So, using habit utilities it is possible to obtain empirically valid allocations between risky and non-risky investment opportunities in financial economics models.

In the case of habit utilities, it is difficult to get a closed solution: Detemple et al. (1992) find a precise solution of optimal consumption, but not a precise solution of optimal portfolio.

Munk (2008) finds a closed solution when interest rate is deterministic and numerical solution using PDE and Monte Carlo simulation when interest rate is stochastic.
more realistic dynamics of interest rate and asset prices. The interest rate dynamics follows differential equation (CIR)

\[ dr_t = \kappa_r (\bar{r} - r_t) dt + \sigma_r \sqrt{r_t} dz_t \] (6)

and the market-price-of-risk process follows differential equation (mean-reverting stock returns)

\[ d\lambda_t = \kappa_\lambda (\bar{\lambda} - \lambda_t) dt + \sigma_\lambda \sqrt{r_t} dz_t. \] (7)

this paper proposes straightforward method to solve Merton (1971)'s problem in the case of habit utilities and financial asset adheres to for example the dynamics of equations (6) and (7)
When markets are complete it is possible solve optimal consumption as a static optimization problem (Karatzas et al. (1987) and Cox and Huang (1989))

When $\lambda_t = \frac{\alpha_t - r_t}{\sigma_t}$ is market-price-of-risk state price density

$$
\zeta_t = e^{\int_0^t r_u du} \xi_t, \quad \xi_t = e^{-\int_0^t \lambda_s dz_s - \frac{1}{2} \int_0^t ||\lambda_s||^2 ds}
$$
Optimal Consumption Choice

Detemple et al. (1992) solve the optimal consumption in the case of power habit utilities:

\[
c(y^*)_t = z_0 e^{-\int_0^t (a-b)dv} + (y^*)^{1/\rho-1} \left[ \phi_t^{1/\rho-1} + \int_0^t be^{-\int_s^t (a-b)du} \phi_s^{1/\rho-1} ds \right]
\]

(9)

where Lagrange coefficient

\[
y = [w_0 - z_0 E \int_0^T e^{\int_0^t (b(v)-a(v))dv} dt]^{\rho-1} [E \int_0^T e^{-\int_0^T r_u du}] \nonumber
\]

\[\times \left[ \phi_t^{1/\rho-1} + \int_0^t be^{-\int_s^t (a-b)du} \phi_s^{1/\rho-1} ds \right]^{1-\rho}
\]

(10)

and

\[
\phi_t = \zeta_t (1 + bE[ \int_t^T e^{\int_t^s (-r(v)-b+a)ds} ds])
\]

(11)
The Idea of Monte Carlo Covariation Method

- Cvitanic etc. (2003) solve the optimal portfolio when utilities are time-separable using Monte Carlo covariation method.
- They start considering an expression

\[ C_t = E\left[ \int_t^T f(r_s, \lambda_t, z_s) ds | \mathcal{F}_t \right] \]  \hspace{1cm} (12)

which satisfies a stochastic differential equation of the type

\[ dC_t = \varphi_t dt + \nu_t dz_t \]  \hspace{1cm} (13)

where \( \varphi_t \) is the drift and \( \nu \) is diffusion coefficient and \( r_s, \lambda_t \) and \( z_s \) as before. The wealth process

\[ dw_t = (\pi_t \mu_t + (w_t - \pi_t)r_t - c_t) dt + \pi_t \sigma_t dz_t \]

Because the diffusion terms of (3) and (13) equal, holds

\[ \nu_t = \pi_t^* \sigma_t \iff \pi_t^* = (\sigma_t)^{-1} \nu_t \]  \hspace{1cm} (14)

If we can solve \( \nu_t \) by simulation it is also possible to solve optimal portfolio choice as a linear transformation of of the wealth process.
The limit

\[ v_t = \lim_{\Delta t \to 0} E\left[ \frac{(C_{t+\Delta t} - C_t)}{\Delta t} | \tilde{F}_t \right] \tag{15} \]

is the foundation of approximation and estimate of \( v_t \) can be computed by

\[ \hat{v}_t = \frac{1}{K} \sum_{i=1}^{K} \frac{(C_{t+\Delta t} - C_t) z_t}{\Delta t} \tag{16} \]

where \( z_t \) is standard normal random variable and \( K \) the total number of simulated paths. The covariation between the optimal wealth process and the uncertainty shocks provides expression for the volatility. For \( v_t \) holds

\[ v_t = \lim_{\Delta t \to 0} E\left[ \frac{(w_{t+\Delta t} - w_t)(z_{t+\Delta t} - z^i_t)}{\Delta t} | \tilde{F}_t \right] \tag{17} \]
I have worked on Monte Carlo covariation method so that it is applicable in the case of habit utilities.

habit coefficients $a$ and $b$ are assumed to be constant

The implementation of method:
1. Solution of optimal consumption path
2. Solution of volatility (as Cvitanic et al. (2003))
3. Solution of optimal portfolio choice as the linear transformation of volatility
power utility form: \( u(\cdot) = \frac{1}{\gamma}(c - h)^\gamma \)

After the solution of Lagrange coefficient, I use an algorithm which, at first, calculates the optimal path of consumption (9). \((\Delta t = T/N, N = 100). In the every step it is necessary a generate pseudo-random value \(z^i_j\) with distribution \(N(0, \Delta t)\) and upgrade the values of interest rate and the values of market-price-of-risk):

\[
c_t = h_0 e^{-\int_0^t (a-b)dv} + (y^*)^{1/\rho-1}[(\zeta_t \eta_t)^{1/\rho-1} + \int_0^t b e^{-\int_s^t (a-b)du} (\zeta_s \eta_s)^{1/\rho-1} ds]
\]  

We get \(c_0+\Delta t, c_0+2\Delta t, \ldots, c_T\) and then the value of wealth process at time \(t + \Delta t\)
The value of wealth process at every point in time is the expected discounted value of future consumption. So

\[
  w_{t+\Delta t} = E \left[ \int_{t+\Delta t}^{T} e^{-\int_{t+\Delta t}^{s} r_u \, ds} \frac{\xi_s}{\xi_{t+\Delta t}} c^*_s \, ds | \mathcal{F}_{t+\Delta t} \right]
\]  

(19)

Now, the estimate of \( v_t \) can be computed similarly than Cvitanic et al. (2003) using 2-tier simulation. At first an estimate for \( w^*_t(z_{1}^i) \) is calculated by

\[
  w^*_t(z_{j}^i) = \frac{1}{M} \sum_{j=1}^{M} \int_{t+\Delta t}^{T} H_{t+\Delta t} c^*_s \, ds.
\]  

(20)

And the volatility of wealth process is solved using

\[
  \hat{v}_t = \frac{1}{K} \sum_{i=1}^{K} \left[ \frac{(w_{t+\Delta t}(z_{j}^i) - w_t)(z_{j}^i)}{\Delta t} \right].
\]  

(21)

where \( z_{j}^i \) is the \( j \):st step of path \( i \) and \( K \) the total number of simulated paths. The covariation between the optimal wealth process and the uncertainty shocks provides expression for the optimal portfolio.
I follow Cvitanic et al. (2003) and Detemple et al. (1999) and assume same values of constants than they: \( \bar{r} = 0.06, \)
\( \sigma_r = 0.0364, \kappa_r = 0.0824, \kappa_\theta = 0.6950, \bar{\theta} = 0.0871, \)
\( \sigma_\theta = 0.21, \sigma_t = 0.2, r_0 = 0.06, \theta_0 = 0.1 \)

**Table:** Optimal portfolio for different parameters a and b and for different values of risk aversion when time horizon T=1.

<table>
<thead>
<tr>
<th>( \pi )</th>
<th>( \gamma = -1 )</th>
<th>( \gamma = -2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>a=0 &amp; b=0</td>
<td>0.243</td>
<td>0.174</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.2</td>
<td>0.209</td>
<td>0.138</td>
</tr>
<tr>
<td>a=0.1 &amp; b=0.3</td>
<td>0.220</td>
<td>0.153</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.3</td>
<td>0.205</td>
<td>0.142</td>
</tr>
<tr>
<td>a=0.2 &amp; b=0.4</td>
<td>0.215</td>
<td>0.134</td>
</tr>
<tr>
<td>a=0.4 &amp; b=0.5</td>
<td>0.199</td>
<td>0.161</td>
</tr>
</tbody>
</table>
It is easy to get solution for case of different values of constants and \( T \)

Using \( K = 50000 \) and \( M = 50 \), I get quite similar size standard deviation than Cvitanic et al. (2003) in time separable case (0.002 when \( K=10000,M=50 \)). Using MATLAB program on standard desktop PC the computational times are from 8 minutes (\( T=1 \)) to about 1 and half hour (\( T=10 \)) and are not substantially longer than in Cvitanic et al. (2003)
This paper shows method for solving the optimal portfolio choice of investor with habit utilities only restriction about the behavior of financial assets are complete markets and the expanded opportunity set has to be Markovian.

In an example a case in which interest rate dynamics is CIR and stock returns are mean-reverting has been considered.
Some references


