Inverse QuickXplain vs. MaxSAT –
A Comparison in Theory and Practice

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Overview

• Motivation
• Preferred Minimal Diagnosis
• Partial Weighted MinUNSAT
• Complexity
• Reductions
Overview

- **Motivation**
- Preferred Minimal Diagnosis
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- Reductions

Motivation
Motivation (cont’d)

• What to do in the unsatisfiable case?
  • Conflict (MUS / Proof / Explanation)
Motivation (cont’d)

• What to do in the unsatisfiable case?
  • Conflict (MUS / Proof / Explanation)
  • Diagnosis (MCS)
    a) Order (Preferred Minimal Diagnosis)
    b) Priorities (Partial Weighted MinUNSAT)

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Minimal Correction Subset (MCS)

- Conjunction of clauses (CNF):

\[
(\neg x \lor \neg y) \quad (y) \quad (a \lor b) \quad (x) \quad (\neg y \lor \neg z) \quad (z)
\]

- MCS: Minimal subset such that the complement is satisfiable

Inverse QuickXplain vs. MaxSAT
Conjunction of clauses (CNF):

\[(\neg x \lor \neg y) (y) (a \lor b) (x) (\neg y \lor \neg z) (z)\]

- MCS: Minimal subset such that the complement is **satisfiable**
- The complement of a **MCS** is a Maximal Satisfiable Subset (**MSS**)

Hard constraints \(\phi_{hard}\), soft constraints \(\phi_{soft} = \{c_1, \ldots, c_m\}\)
Preferred Minimal Diagnosis

- Strict total order $<$ on clause set $\phi = \{c_1, \ldots, c_m\}$:
  \[ c_1 < \cdots < c_m \]

- Lexicographic preference: $\psi_1, \psi_2 \subseteq \phi$
  $\psi_1 <_{lex} \psi_2$ iff $\exists 1 \leq k \leq m : c_k \in \psi_1 \setminus \psi_2 \wedge \psi_1 \cap \{c_1, \ldots, c_{k-1}\} = \psi_2 \cap \{c_1, \ldots, c_{k-1}\}$

- Anti-lexicographic preference: $\psi_1, \psi_2 \subseteq \phi$
  $\psi_1 <_{antilex} \psi_2$ iff $\exists 1 \leq k \leq m : c_k \in \psi_2 \setminus \psi_1 \wedge \psi_1 \cap \{c_{k+1}, \ldots, c_m\} = \psi_2 \cap \{c_{k+1}, \ldots, c_m\}$

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Preferred Minimal Diagnosis (cont’d)

- L-Preferred MSS w.r.t. order $<$
  \[ c_1 < c_2 < c_3 < \cdots < c_{m-2} < c_{m-1} < c_m \]
  Keep most preferred clauses

- A-Preferred MCS (Preferred Minimal Diagnosis) w.r.t. order $<^{-1}$
  \[ c_1 > c_2 > c_3 > \cdots > c_{m-2} > c_{m-1} > c_m \]
  Discard least preferred clauses
Preferred Minimal Diagnosis (cont’d)

- L-Preferred MSS w.r.t. order <
  \[ c_1 < c_2 < c_3 < \cdots < c_{m-2} < c_{m-1} < c_m \]
  Keep most preferred clauses

- A-Preferred MCS (Preferred Minimal Diagnosis) w.r.t. order \(<^-1\)
  \[ c_1 > c_2 > c_3 > \cdots > c_{m-2} > c_{m-1} > c_m \]
  Discard least preferred clauses

- Complement of L-Preferred MSS w.r.t. < is the A-Preferred MCS w.r.t. \(<^-1\)

Preferred Minimal Diagnosis – Algorithms

- Linear Search:
  \[ (\neg x \lor \neg y) < (y) < (a \lor b) < (x) < (\neg y \lor \neg z) < (z) \]

- Worst case number of SAT calls: \(O(m)\)
Inverse QuickXPlain [FS2010]:

(¬x ∨ ¬y) < (y) < (a ∨ b) < (x) < (¬y ∨ ¬z) < (z)

Preferred Minimal Diagnosis – Algorithms

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?
Preferred Minimal Diagnosis – Algorithms

• Inverse QuickXPlain [FS2010]:

\[ \neg x \lor \neg y < y < a \lor b < x < \neg y \lor \neg z < z \]

Preferred Minimal Diagnosis – Algorithms

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Preferred Minimal Diagnosis – Algorithms

- Inverse QuickXPlain [FS2010]:
  \[
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  \]

\[
(\neg x \lor \neg y) < (y) < (a \lor b) < (x) < (\neg y \lor \neg z) < (z)
\]

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Inverse QuickXplain vs. MaxSAT
Preferred Minimal Diagnosis – Algorithms

• Inverse QuickXPlain [FS2010]:

$$\neg x \lor \neg y < y < a \lor b < x < \neg y \lor \neg z < z$$
Preferred Minimal Diagnosis – Algorithms

• Inverse QuickXPlain [FS2010]:

\[
\neg x \lor \neg y < (y) < (a \lor b) < (x) < \neg y \lor \neg z < (z)
\]

• Let $d$ be the minimal diagnosis set size
• Worst case number of SAT calls: \( O \left( 2d \cdot \left( \log_2 \left( \frac{m}{d} \right) + 1 \right) \right) \)

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Partial Weighted MinUNSAT

• Conjunction of Clauses (CNF):

\[(\neg x \lor \neg y) \quad (y) \quad (a \lor b) \quad (x) \quad (\neg y \lor \neg z) \quad (z)\]

\[
\begin{array}{cccc}
5 & 2 & 2 & 2 \\
\end{array}
\]

\[\phi_{\text{hard}}\]

• **Question**: *Minimum* sum of weights of *unsatisfied* soft clauses?

• **Answer**: \[2 + 2 = 4\]
Partial Weighted MinUNSAT

- Conjunction of Clauses (CNF):

\[
(\neg x \lor \neg y) \quad (y) \quad (a \lor b) \quad (x) \quad (\neg y \lor \neg z) \quad (z)
\]

- **Question**: Minimum sum of weights of unsatisfied soft clauses?
- **Answer**: \(2 + 2 = 4\)

- Complement of P.W.MinUNSAT is P.W.MaxSAT

Partial Weighted MinUNSAT – Algorithms

- Binary Search [HMM2011]
  - Range \(lb = 0\) to \(ub = \sum_i^m w_i\)
  - Add fresh blocking variable to each clause
  - Add PBC constraint to narrow search space
  - Worst case number of SAT calls: \(O(\log_2(\sum_i^m w_i))\)
Partial Weighted MinUNSAT – Algorithms

- Binary Search [HMM2011]
  - Range $lb = 0$ to $ub = \sum_{i}^{m} w_i$
  - Add fresh blocking variable to each clause
  - Add PBC constraint to narrow search space
  - Worst case number of SAT calls: $O(\log_2(\sum_{i}^{m} w_i))$
- Branch and Bound [Kue2012]
- Unsatisfiable Core Guided [FM2006, ABL2009]
  - Iterative SAT calls
  - Exploit provided unsat. core
  - Worst case number of SAT calls: $O(\sum_{i}^{m} w_i)$
  - Open topic: Relationship of unsat. core and SAT calls
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Complexity

- $FP^{NP}$: Function problems solvable with a polynomial number of NP-oracle calls
Complexity

- **FP\(^{NP}\):** Function problems solvable with a polynomial number of NP-oracle calls

- Partial Weighted MinUNSAT is …
  - in FP\(^{NP}\) (Binary Search)
  - FP\(^{NP}\)-hard [Pap1994]

\[ \text{FP}^{NP}\text{-complete} \]
Complexity

- \( \text{FP}^\text{NP} \): Function problems solvable with a polynomial number of \( \text{NP} \)-oracle calls

- Partial Weighted MinUNSAT is ...  
  - in \( \text{FP}^\text{NP} \) (Binary Search)  
  - \( \text{FP}^\text{NP} \)-hard [Pap1994]  

- A-Preferred MCS is ...  
  - in \( \text{FP}^\text{NP} \) (Linear Search)  
  - \( \text{FP}^\text{NP} \)-hard (Proven in this work)

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Complexity

- Theorem: A-Preferred MCS is \( \text{FP}^\text{NP} \)-hard
- Proof (sketch):
  - Problem: Maximum Satisfying Assignment (MSA)
    Input: Boolean formula \( \phi \) over variables \( x_1, \ldots, x_n \)
    Find satisfying assignment with lex. maximum of word \( x_1 \cdots x_n \in \{0,1\}^n \)
  - MSA is \( \text{FP}^\text{NP} \)-hard [Kre1988]
  - Reduce MSA polynomial to A-Preferred MCS:
    \( \phi_{\text{hard}} = \text{Tseitin}(\phi), \phi_{\text{soft}} = \{x_1, \ldots, x_n\}, \quad x_1 < \cdots < x_n \)
  - Solve A-Preferred MCS w.r.t. \( <^{-1} \)
  - Complement (L-Preferred MSS) is solution to MSA problem
Complexity

- \( \text{FP}^{\text{NP}} \): Function problems solvable with a polynomial number of NP-oracle calls

- Partial Weighted MinUNSAT is ...
  - in \( \text{FP}^{\text{NP}} \) (Binary Search)
  - \( \text{FP}^{\text{NP}} \)-hard \([\text{Pap1994}]\)

- A-Preferred MCS is ...
  - in \( \text{FP}^{\text{NP}} \) (Linear Search)
  - \( \text{FP}^{\text{NP}} \)-hard (Proven in this work)

- **Result: Same Complexity**

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- **Reductions**
A-Preferred MCS to P.W. MinUNSAT

- **A-Preferred MCS**: \( \phi_{hard}, \ \phi_{soft} = c_1 < \cdots < c_m \)

- **Hard clauses**: \( \phi_{hard} \)
- **Soft clauses**: \( \{c_1, \ldots, c_m\} \)
- **Weight \( i \)**: \((\sum_{j=i+1}^{m} w_j) + 1\)

**Drawback**: Exponential growth of weights:
\[
\left( \sum_{j=i+1}^{m} w_j \right) + 1 = 2^{m-i} \in O(2^m)
\]

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P.W. MinUNSAT to A-Preferred MCS

- **P.W. MinUNSAT**: \( \phi_{hard}, \phi_{soft} = \{c_1, \ldots, c_m\}, w_1, \ldots, w_m \)

- **Add to hard clauses**: \( \phi_{hard} \)
- **Add to hard clauses**: \( s_i \rightarrow c_i \)
- **Build sum**:
\[
\sum_{i=1}^{m} w_i \cdot s_i = a_l \cdot 2^l + \cdots + a_0 \cdot 2^0
\]
- **Add to hard clauses**: Adder-Network
  - **Input variables**: \( s_1, \ldots, s_m \)
  - **Output variables**: \( a_l, \ldots, a_0 \)
  - **Strict total order**: \( a_l < \cdots < a_0 \)
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Summary

- Motivation
  - Handle unsatisfiable case
- Preferred Minimal Diagnosis
  - Linear Search, Inverse QuickXPlain
  - \( \text{FP}^{\text{NP}} \)-complete
- Partial Weighted MinUNSAT
  - Binary Search, Unsat. Core Guided
  - \( \text{FP}^{\text{NP}} \)-complete
- Reductions

Thank you for your attention
Bibliography

- [FM2006] Fu, Malik. **On Solving the partial MaxSAT Problem.** SAT 2006
- [Pap1994] Papadimitrou. **Computational Complexity.** Addison-Wesley, 1994