# Inverse QuickXplain vs. MaxSAT –
## A Comparison in Theory and Practice

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# Overview

- Motivation
- Preferred Minimal Diagnosis
- Partial Weighted MinUNSAT
- Complexity
- Reductions

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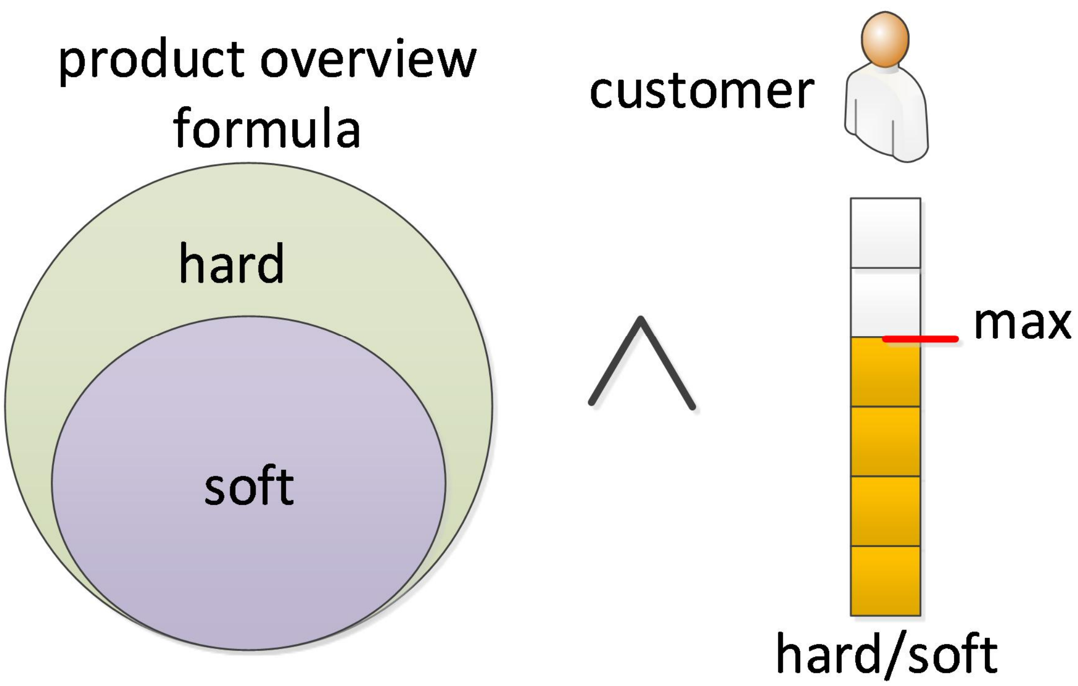
# Motivation



## Motivation (cont'd)

- What to do in the unsatisfiable case?



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  - Conflict (MUS / Proof / Explanation)

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- What to do in the unsatisfiable case?
  - Conflict (MUS / Proof / Explanation)
  - Diagnosis (MCS)
    a) Order (Preferred Minimal Diagnosis)
    b) Priorities (Partial Weighted MinUNSAT)



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# Minimal Correction Subset (MCS)

- Conjunction of clauses (CNF):

$(\neg x \lor \neg y)$ $(y)$ $(a \lor b)$ $(x)$ $(\neg y \lor \neg z)$ $(z)$

- MCS: Minimal subset such that
  the complement is satisfiable



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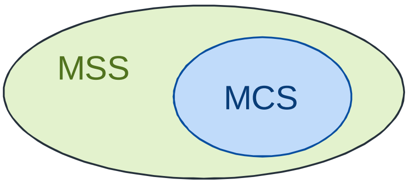




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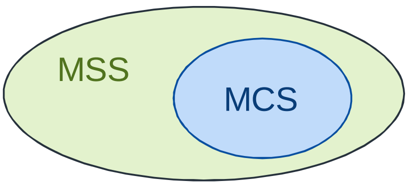


- Hard constraints $\phi_{hard}$, soft constraints $\phi_{soft} = \{c_1, \dots, c_m\}$

# Preferred Minimal Diagnosis

- Strict total order $<$ on clause set $\phi = \{c_1, \ldots, c_m\}$:
$$c_1 < \cdots < c_m$$

- Lexicographic preference: $\psi_1, \psi_2 \subseteq \phi$
$\psi_1 <_{lex} \psi_2$ iff $\exists 1 \leq k \leq m: c_k \in \psi_1 \setminus \psi_2 \wedge$
$\psi_1 \cap \{c_1, \ldots, c_{k-1}\} = \psi_2 \cap \{c_1, \ldots, c_{k-1}\}$

- Anti-lexicographic preference: $\psi_1, \psi_2 \subseteq \phi$
$\psi_1 <_{antilex} \psi_2$ iff $\exists 1 \leq k \leq m: c_k \in \psi_2 \setminus \psi_1 \wedge$
$\psi_1 \cap \{c_{k+1}, \ldots, c_m\} = \psi_2 \cap \{c_{k+1}, \ldots, c_m\}$



# Preferred Minimal Diagnosis (cont'd)

- L-Preferred MSS w.r.t. order $<$
$$c_1 < c_2 < c_3 < \cdots < c_{m-2} < c_{m-1} < c_m$$
Keep most preferred clauses

- A-Preferred MCS (Preferred Minimal Diagnosis) w.r.t. order $<^{-1}$
$$c_1 > c_2 > c_3 > \cdots > c_{m-2} > c_{m-1} > c_m$$
Discard least preferred clauses

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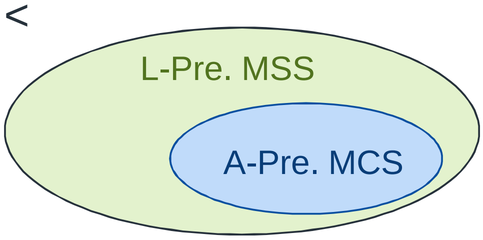
- Complement of L-Preferred MSS w.r.t. $<$ is the A-Preferred MCS w.r.t. $<^{-1}$






# Preferred Minimal Diagnosis – Algorithms

- Linear Search:

$$(\neg x \vee \neg y) < (y) < (a \vee b) < (x) < (\neg y \vee \neg z) < (z)$$

- Worst case number of SAT calls: $O(m)$

# Preferred Minimal Diagnosis – Algorithms

- Inverse QuickXPlain [FS2010]:

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- Let d be the minimal diagnosis set size
- Worst case number of SAT calls: $O\left(2d \cdot \left(\log_2\left(\frac{m}{d}\right) + 1\right)\right)$



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# Partial Weighted MinUNSAT

- Conjunction of Clauses (CNF):

| $(\neg x \vee \neg y)$ | $(y)$ | $(a \vee b)$ | $(x)$ | $(\neg y \vee \neg z)$ | $(z)$ |
|---|---|---|---|---|---|
| 5 | 2 | 2 | 2 | $\phi_{hard}$ | |

- **Question**: *Minimum* sum of weights of unsatisfied soft clauses?



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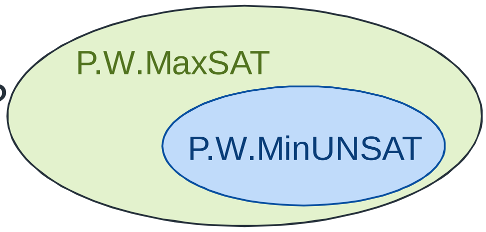


- Complement of P.W.MinUNSAT is P.W.MaxSAT



# Partial Weighted MinUNSAT – Algorithms

- Binary Search [HMM2011]
  - Range $lb = 0$ to $ub = \sum_i^m w_i$
  - Add fresh blocking variable to each clause
  - Add PBC constraint to narrow search space
  - Worst case number of SAT calls: $O(\log_2(\sum_i^m w_i))$

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  - Worst case number of SAT calls: $O(\log_2(\sum_i^m w_i))$
- Branch and Bound [Kue2012]
- Unsatisfiable Core Guided [FM2006, ABL2009]
  - Iterative SAT calls
  - Exploit provided unsat. core
  - Worst case number of SAT calls: $O(\sum_i^m w_i)$
  - Open topic: Relationship of unsat. core and SAT calls

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  - in $FP^{NP}$ (Binary Search)
  - $FP^{NP}$–hard [Pap1994]

  } $FP^{NP}$-complete



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  - $FP^{NP}$–hard (Open [MS2014])

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  } $FP^{NP}$-complete



# Complexity

- Theorem: A-Preferred MCS is $FP^{NP}$–hard
- Proof (sketch):
  - Problem: *Maximum Satisfying Assignment* (MSA)
    Input: Boolean formula $\phi$ over variables $x_1, \dots, x_n$
    Find satisfying assignment with lex. maximum of word
    $$x_1 \cdots x_n \in \{0, 1\}^n$$
  - MSA is $FP^{NP}$-hard [Kre1988]
  - Reduce MSA polynomial to A-Preferred MCS:
    $\phi_{hard} = \text{Tseitin}(\phi), \phi_{soft} = \{x_1, \dots, x_n\}, \quad x_1 < \cdots < x_n$
  - Solve A-Preferred MCS w.r.t. $<^{-1}$
  - Complement (L-Preferred MSS) is solution to MSA problem

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- **Result: Same Complexity**



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# A-Preferred MCS to P.W. MinUNSAT

- A-Preferred MCS: $\phi_{hard}, \quad \phi_{soft} = c_1 < \cdots < c_m$

- Hard clauses: $\phi_{hard}$
- Soft clauses: $\{c_1, \ldots, c_m\}$
- Weight $i$: $\left(\sum_{j=i+1}^{m} w_j\right) + 1$

- Drawback: Exponential growth of weights:

$$\left(\sum_{j=i+1}^{m} w_j\right) + 1 = 2^{m-i} \in O(2^m)$$



# P.W. MinUNSAT to A-Preferred MCS

- P.W. MinUNSAT: $\phi_{hard}, \phi_{soft} = \{c_1, \ldots, c_m\}, w_1, \ldots, w_m$

- Add to hard clauses: $\phi_{hard}$
- Add to hard clauses: $s_i \rightarrow c_i$
- Build sum:

$$\sum_{i=1}^{m} w_i \cdot s_i = a_l \cdot 2^l + \cdots + a_0 \cdot 2^0$$

- Add to hard clauses: Adder-Network
  - Input variables: $s_1, \ldots, s_m$
  - Output variables: $a_l, \ldots, a_0$
- Strict total order: $a_l < \cdots < a_0$

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# Summary

- Motivation
  - Handle unsatisfiable case
- Preferred Minimal Diagnosis
  - Linear Search, Inverse QuickXPlain
  - $FP^{NP}$-complete
- Partial Weighted MinUNSAT
  - Binary Search, Unsat. Core Guided
  - $FP^{NP}$-complete
- Reductions

**Thank you for your attention**

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