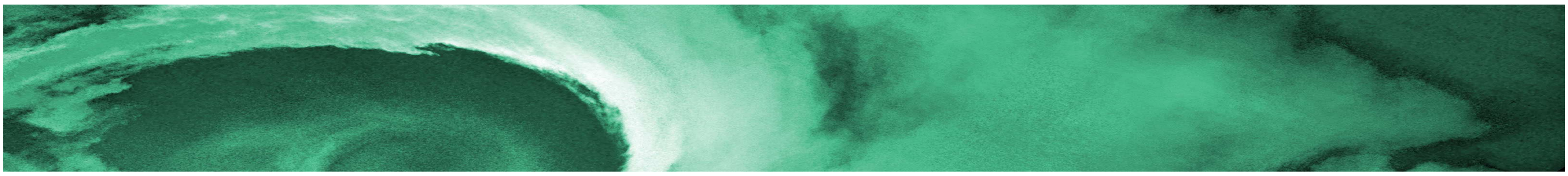

Generalised Entropy Definition Applied to Turbulent Space Plasmas

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Introduction

Entropy is a fundamental quantity describing the number of possible states of a system. According to the second law of thermodynamics, on a global scale entropy can only increase as the system evolves, and a number of processes such as turbulence contribute to its growth. But how to define entropy in turbulent, collisionless plasmas?

Tsallis entropy

There are several definitions for entropy of which Gibbs-Boltzmann (GB) is the most commonly used. Usually we make the assumption of adiabatic perfect gas so that the Gibbs-Boltzmann entropy can be written as

$$S_{BG} \equiv -k \sum_i p_i \ln p_i = c_V \ln(P/\rho^\gamma), \tag{1}$$

where c_V is the specific heat at constant volume, P the plasma pressure, ρ the plasma density, and γ the adiabatic index. But what if we used a generalised entropy definition: the q-entropy proposed by Tsallis [1]

$$S_q \equiv k\frac{1-\sum_i p_i^q}{q-1} = k\sum_i p_i \ln_q(1/p_i) \underset{q\to 1}{\longrightarrow} S_{BG}. \tag{2}$$

The entropic index q characterises the the degree of non-extensivity (non-locality) of the system.

This non-extensive entropy seems better suited for describing space plasmas: its theoretical formulation allows the system to have long range interactions and memory effects. Moreover, while GB entropy produces a Maxwellian distribution in velocity space, the equilibrium distribution for Tsallis entropy is a kappa distribution [2,3] similar to the ones observed, e.g., in the solar wind:

$$\begin{aligned} F = \frac{(1-q)^{3/2}}{\pi^{3/2} v_{th}^3}&\left[\frac{\Gamma(1/(1-q)-3/2)}{\Gamma(1/(1-q))} + \frac{\Gamma(1/(1-q)+1)}{\Gamma(1/(1-q)+5/2)}\right]^{-1} \\ &\times\left\{\left[1+(1-q)\frac{(\bar v-\bar v_s)^2}{v_{th}^2}\right]^{\frac{-1}{1-q}} \qquad \text{halo} \right. \\ &\left. +\left[1-(1-q)\frac{\bar v^2}{v_{th}^2}\right]^{\frac{1}{1-q}}\right\}. \qquad \text{core} \end{aligned} \tag{3}$$

Here v_{th} is the thermal speed, $\bar v_s$ the shift in velocity space, and $-1 < q \le 1$, with condition $q_h = q_c = q$ for particle conservation and a single Maxwellian in the limit $q \to 1$. Note that the core part of the distribution is only used at $v^2 \le v_{th}^2/(1-q)$.

Data and Instruments

We use the generalised definition (2) to analyse proton and electron distribution data from the Wind satellite in order to calculate the evolution of entropy in the solar wind plasma. The 3D plasma analyzer experiment [4] measures the full three-dimensional distribution of ions and electrons at energies 3 eV to 30 keV.

Method of Maximum Likelihood

Given a statistical model $f_{\bar Y}(\bar y;\bar\theta)$ containing an unknown parameter $\bar\theta$ for a random variable $\bar Y$, the likelihood function associated with a set of observations $\bar y$ is

$$\mathcal{L}(\bar\theta) = f_{\bar Y}(\bar y;\bar\theta) \stackrel{\perp\!\!\!\perp}{=} \prod_{i=1}^{n} f_{Y_i}(y_i;\bar\theta). \tag{4}$$

Equivalently, we can use the logarithmic likelihood function

$$\ell(\bar\theta) = \ln\mathcal{L}(\bar\theta) \stackrel{\perp\!\!\!\perp}{=} \sum_{i=1}^{n} \ln f_{Y_i}(y_i;\bar\theta). \tag{5}$$

The maximum likelihood estimate for parameter $\bar\theta$ is given the set of observations $\bar y$

$$\hat{\bar\theta} = \hat{\bar\theta}(\bar y) = \max_{\bar\theta} \ell(\bar\theta). \tag{6}$$

For our study, $\bar\theta = (q, v_{th}, \bar v_s)$ and the statistical model is given by (3). The q-entropy can then be calculated using

$$S_q = k\frac{1-\int[\hat v_{th}^3 F(\bar v;\hat q,\hat v_{th},\hat{\bar v}_s)]^{\hat q} d(\bar v/\hat v_{th}^3)}{q-1}. \tag{7}$$

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