Repeated games

Lecture 10

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Repeated games

- In a repeated game there is a fixed stage game, say prisoners' dilemma, that is played over and over again.
- There is a big difference whether the stage game is played finitely many times or infinitely many times.
- To study repeated games it is necessary to agree on the way the players evaluate the pay-offs.
- The most common, albeit by no means the only, practice is to postulate that after each round of the stage game the players receive the stage game pay-offs, and that they evaluate the stream, finite or infinite, of expected pay-offs as separable over time and discounted.

Repeated games

- Consider an infinitely repeated prisoners' dilemma

	C	D
C	$2,2$	$0,3$
D	$3,0$	$1,1$

- Choosing (D,D) in the first round and then (C,C) in each successive round players' pay-offs are

$$1+\sum_{t=1}^{\infty}\delta^t 2 = 1+\frac{\delta}{1-\delta}2$$

- This is the discounted 'life-time' utility from the above play.
- It is convenient to consider the average utility, or per period utility, that generates this life-time utility.
- This is got by multiplying the life-time utility by $1-\delta$.
- Above this would yield $1-\delta+2\delta = 1+\delta$.

Repeated games

- The Nash-equilibrium pay-offs of strategic games are typically not on the Pareto frontier.
- One of the main questions in repeated games is whether there are equilibria whose periodic outcomes are on the Pareto frontier of the stage game.
- This is most challenging in the prisoners' dilemma, and for this reason it is the most common stage game in applications.
- A repeated game is an extensive form game, and its formal definition is straightforward.
- It is, though, quite complicated.
- Consider a 2x2 game which is played three times.
- The number of strategies in this game is got as follows.
- In the first stage there are 2 ways of making a choice.
- In the second stage a strategy has to specify what to do after each possible history.

- As there are four histories and two actions the number of possible ways of making a choice is 2^4.
- In the third stage there are $4 \cdot 4 = 16$ histories and two actions.
- Consequently, there are 2^{16} ways of making a choice.
- Thus, the number of strategies is $2 \cdot 2^4 \cdot 2^{16} = 2^{21}$.

- Consider first prisoners' dilemma that is played T times (presumably repeated T-1 times).
- An outcome path is any end node.
- The only Nash-equilibrium outcome path of this game is D in each round.
- First, it is a Nash-equilbrium to play D in each round.
- Fix a Nash-equilibrium and let t be the last stage such that at least one player chooses C.
- But deviating to D in stage t increases the deviating player's pay-off, and changes nothing for the rest of the game.

Repeated games

- Consider the following stage game repeated once, i.e., played twice

	L	C	R
T	$6,6$	$0,7$	$1,2$
M	$7,0$	$1,1$	$2,0$
B	$2,1$	$0,1$	$3,3$

- A strategy where player-1 chooses T in the first round, and B in the second round if the history is (T,L) and M if the history is something else, and player-2 chooses L in the first round, and R in the second round if the history is (T,L) and C if the history is something else, constitutes an equilibrium that supports the co-operative outcome.
- This works if there are several stage game equilibria.
- The worst of them can be used to threaten bad behaviour.
- The threat is credible since playing an equilibrium is credible.

Repeated games

- Let us return to the prisoners' dilemma, and consider the infinitely repeated version.
- There are several well-known strategies that we analyse next.
- The grim-trigger strategy is such that a player chooses C in the first period, and continues to do so unless his/her opponent chooses D.
- After D by the opponent s/he chooses D forever.
- This is a Nash-equilibrium if the discount factor is high enough.
- On the Nash-equilibrium outcome path C is played each period and the average pay-off is 2.
- If one player deviates and chooses D in the first period then his/her opponent chooses D forever from the second period on, and the deviating player's optimal deviation is D forever.

- His/her average pay-off is$(1-\delta)\left(3+\frac{\delta}{1-\delta}\right)=3-2\delta$.
- This is less than 2 iff $\delta \geq \frac{1}{2}$.
- In the grim-trigger strategy the 'punishment'-phase following a deviation is infinitely long.
- In a straightforward fashion it is possible to determine when a strategy with a finite punishment-phase constitutes a Nash-equilibrium.

Repeated games

- The strategy called tit-for-tat postulates that in the first stage a player chooses C and in each consequent stage s/he chooses what his/her opponent chose in the previous stage.
- If this is a Nash-equilibrium then on the outcome path C is played in each period.
- If one player deviates to D in period t then the other player chooses D in period $t+1$.
- Thus, the deviator has two choices: Either to choose D forever or to revert to C in period t+1 but then s/he is in the same situation as in period t.
- Consequently, the optimal deviation is D forever or to alternate between D and C.
- Choosing D forever yields $(1-\delta)\left(3+\frac{\delta}{1-\delta}\right)$, while alternating yields $(1-\delta)\frac{3}{1-\delta^2}$.
- The maximun of the above pay-offs is less than 2 iff $\delta \geq \frac{1}{2}$.

Repeated games

- It is clear that any convex combination of the average pay-offs can be generated (to a required degree) by some strategy combination if the discount factor is high enough.
- For instance, $1.45 = \frac{145}{100} = \frac{29}{20} = \frac{9 \cdot 3 + 1 \cdot 2 + 10 \cdot 0}{20}$ for player 1 can be got on the outcome path

$$((D, C), \ldots, (D, C), (C, C), (C, D), \ldots, (C, D))$$

where the first string is of lenght 9 and the last string is of length 10.

- Repeating this string indefinitely (supported by the threat that deviation results in perpetual D) generates the desired pay-off.
- The feasible average pay-offs is a convex set, the convex combination of the possible pay-offs of the stage game.
- Picking any point from there one can read from the weights which outcome path generates that pay-off.

Repeated games

- Not all Nash-equilibria are SPE.
- There is a simple test for perfectness.

One-deviation property: No player can increase his/her pay-off by changing his/her action in any subgame which s/he starts given the other players' strategies and the rest of his/her strategy.

Repeated games

Theorem

A strategy profile is a SPE iff it satisfies the one-deviation property.

- The grim-trigger strategy is not a SPE since in a subgame following a deviation by a player this player is supposed to choose C.
- Changing the strategy so that it postulates the choice D whenever the history contains choices different from C restores subgame perfectness.

Theorem

(Subgame perfect folk theorem). Let $(x_1, x_2) > (1, 1)$ be any average feasible pay-off in the infinitely repeated prisoners' dilemma. For sufficiently high value of the discount factor there exists a subgame perfect Nash-equilibrium that generates average pay-offs (x_1, x_2).

- Basic wisdom: The greater the possible punishment the easier it is to sustain co-operative behaviour.
- In the twice played game above co-operation was supported by the threat to play a bad Nash-equilibrium in the last stage.
- But in general the threats need not be Nash-equilibria of the stage game.