Non-co-operative games

Lecture 2

October 29, 2015

- Next we shall see many examples but let us first look at Nash-equilibrium from another point of view.

Best-response or best-reply functions.

- We introduced Nash-equilibrium as a profile of actions (an action for each player) such that no one player has an incentive to choose a different action (provided that others stick to their choice).
- It is clear that at a Nash-equilibrium (point) each player's choice is a best response to the other players' choices.
- This may lead one to expect that Nash-equilibrium is a fixed point of the players' best-response functions (or correspondences to be precise).

Non-co-operative games

Definition

In a normal form game $\Gamma = \left\{N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\right\}$ player i's best-response function is defined as

$$B_i(a_i^*, a_{-i}) = \left\{a_i \in A_i : u_i\left(a_i, a_{-i}\right) \geq u_i\left(a_i', a_{-i}\right)\right\}$$

for all $a_i' \in A_i$.

Now Nash-equilibrium can be defined as follows.

Definition

In a normal form game $\Gamma = \left\{ N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \right\}$ an action profile $a^* \in \prod_{i=1}^{n} A_i$ is a Nash-equilibrium iff $a_i^* \in B_i(a^*)$ for all $i \in N$.

- Next we go through a number of examples

Example

Cournot-competition.
Consider a standard linear inverse demand $p = 1 - q$. In a Cournot-model there are two firms and each chooses how much to offer for sale simultaneously. Assume that marginal costs of production are zero. Assume that firm 2 produces q_2. Firm 1's best-response is given by

$$q_1(q_2) = \arg\max_{q_1} (1 - q_1 - q_2)\, q_1$$

.

Example

This can be found by taking the first order condition

$$q_1(q_2) = \frac{1 - q_2}{2}$$

As the situation is symmetric firm 2's best-response is evidently given by

$$q_2(q_1) = \frac{1 - q_1}{2}$$

Solving these gives a symmetric Nash-equilibrium $(1/3, 1/3)$. This, however, is not the only Nash-equilibrium of the model. The other equilibria, symmetric and non-symmetric, are most easily found by thinking about best-response functions.

Example

All action profiles where $q_1 \geq 1$ and $q_2 \in (1, \infty)$, or $q_2 \geq 1$ and $q_1 \in (1, \infty)$ are also Nash-equilibria. Try to figure out what is their relation to dominance! When there are n identical firms each of them produces $q = \frac{1-c}{n+1}$ in a symmetric equilibrium where marginal costs are c. Total production is $Q = \frac{n}{n+1}(1-c)$ and equilibrium price $p = \frac{1}{n} + \frac{n}{n+1}c$. When n grows without bound one gets the perfect competition outcome.

Example

Bertrand-competition.
The situation is the same as in the Cournot-competition except that the firms choose prices; whatever prices they choose they are committed to serving the realised demand. The consumers buy from the firms with the lowest price, and in the case of equal prices they divide themselves evenly between the firms. Now a firm *i*'s best response-function is as follows

$$B_i(p_j) = \begin{cases} \{p_i : p_i > p_j\} \text{ if } p_j < 0 \\ \{p_i : p_i \geq p_j\} \text{ if } p_j = 0 \\ \varnothing \text{ if } 0 < p_j \leq p \\ \{p^m\} \text{ if } p^m < p_j \end{cases}$$

where p^m indicates the monopoly price.

Example

When firm j chooses a price between the monopoly price and the marginal cost there is no best-response; firm i would like to choose the highest price that is still less than p_j but such does not exist. Graphing the best-response functions it is immediate that the symmetric Nash-equilibrium is $(0, 0)$ where both firms choose price zero (more generally the price that equals the constant marginal cost).

Example

Auctions.
Let us consider so called second-price sealed-bid auction.
There are $n \geq 2$ bidders, and an indivisible object for sale.
Player i has valuation v_i for the object, and if s/he gets it at price p his/her utility is $v_i - p$.
Players' action sets are positive real number from which they choose their bids b_i.
The rules are such that the highest bidder wins, and pays the second highest bid (if there are draws some known rule is applied).
Change the names of the bidders so that the order of the valuations is $v_1 > v_2 > ... > v_n$.

Example

This game is remarkable in that it has a Nash-equilibrium in dominant strategies.
Bidding one's own valuation is a dominant strategy.
The highest valuation bidder cannot affect the price by bidding higher.
If s/he bids less s/he might not get the object.
For any other player a bid higher than his/her valuation might make him/her the winner: bad thing.
Bidding lower would not change his/her pay-off.
This is a situation where revealing one's true preferences is a dominant strategy equilibrium.
There are other equilibria.
If $n = 3$ and the valuations are $v_1 = 10$, $v_2 = 6$ and $v_3 = 2$ the following is a Nash-equilibrium: $(b_1 = 3, b_2 = 97, b_3 = 5)$.

Example

All-pay auctions.

This is a game where the highest bidder wins and all bidders pay their bid.

It can be used to model political lobbying, or rent seeking, where n players invest in, say, bribing a politician who has a right to grant a monopoly or some prize.

Let us assume that the probability of winning the price for player i is given by $p_i = \frac{b_i}{\sum_{k=1}^n b_k}$ where the b_i is the bribe by player i.

If the value of the monopoly right is V then player i's objective is

$$\max_{b_i} p_i V - b_i$$

which is equivalent to

$$\max_{b_i} \frac{b_i}{\sum_{k=1}^n b_k} V - b_i$$

Example

The first order condition is

$$\frac{\sum_{k=1}^{n} b_k - b_i}{\left(\sum_{k=1}^{n} b_k\right)^2} V - 1 = 0$$

Focussing on a symmetric Nash-equilibrium we can postulate that $b_1 = b_2 = \ldots = b_n = b$, and inserting this to the first order condition yields

$$\frac{(n-1)b}{n^2 b^2} V - 1 = 0$$

from which we get

$$b^N = \frac{n-1}{n^2} V$$

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Example

The total expenditure, pure waste if the politician's utility is ignored, is $nb^N = \frac{n-1}{n} V$.
If there are many bidders or lobbyists almost all of the value is wasted in the rent seeking activity.

Example

Guess the 2/3 average.

In this game n people write a number between zero and 100 on a sheet of paper.

The winner is the person whose number is closest to the 2/3 times the average of the numbers.

All the numbers from 67 to 100 are strictly dominated.

Once everyone understands this they realise that the maximum number anyone writes on the paper is at most 67.

But now the situation is the same as before, and all the number from $(2/3)67$ to 67 are strictly dominated.

Continuing this way iteratively eliminating strictly dominated actions we find the unique Nash-equilibrium of the game $(0, 0, \ldots, 0)$.

Example

Cake division.
Two players have to divide a cake.
Both state simultaneously what is the share they want.
If the shares sum to at most unity the cake is divided accordingly.
If the shares exceed unity neither player gets anything.
Any $(x, 1-x)$, $x \in [0,1]$, where x is the share of player 1 constitutes a Nash-equilibrium.
There are others!