Non-co-operative game theory

Lecture 3

October 29, 2015

Mixed strategies

- In many games there are no Nash-equilibria in pure strategies.
- An example is Matching Pennies game below.

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

Mixed strategies

- The solution to this problem involves extending the action spaces of the players to include probability distributions.
- When this extension is made we call the players' action spaces strategy spaces.
- Instead of choosing single actions the players are allowed to choose probability distributions over the original actions.
- The probability distributions are then called **mixed strategies**.
- In the above example players would choose probability distributions $(p, 1-p)$ where p is the probability of choosing action H and $1-p$ is the probability of choosing action T.

Mixed strategies

- Considering mixed strategies as objects of choice presents some problems.
- If a player expects his/her opponent to use a mixed strategy how should s/he evaluate the utility that a particular action gives him/her?
- From the player's point of view s/he is participating in a lottery/gamble, and the situation is like that of a decision maker under uncertainty.
- We know that decision making under uncertainty can be handled with relative ease if the decision maker has von Neumann-Morgenstern preferences, i.e., if his/her preferences have a utility representation in the expected utility form.
- To remind, if a decision maker has von Neumann-Morgenstern type preferences s/he evaluates the expected utility of a lottery q on a countable set A by

$$U(q) = \sum_{a \in A} q(a)u(a)$$

Mixed strategies

Definition

Consider a normal form game $\Gamma = \left\{N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N}\right\}$.
Its mixed extension is a normal form game where each player's action set is $S_i = \left\{p : \int_{A_i} dp = 1\right\}$.

Definition

Consider a normal form game $\Gamma = \left\{N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N}\right\}$ and its mixed extension
$\Gamma^{me} = \left\{N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}\right\}$.
A Nash-equilibrium is a vector of strategies $s^* = (s_1^*, s_2^*, ..., s_n^*)$ such that for all players $i \in N$ $u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)$ for all $s_i \in S_i$.

Mixed strategies

- Notice that in the definition the Nash-equilibrium applies both the to normal form game and its mixed extension; in the sequel we do not make any difference between the two, and when one is looking for Nash-equilibria it is understood that one is looking for equilibria both in pure and mixed strategies.
- The mixed extension makes it possible to show Nash-equilibrium existence in a large class of games.

Theorem

Every finite normal form game has a mixed strategy equilibrium.

Mixed strategies

- To find mixed Nash-equilibria in simple games it is useful to consider a 2x2-game where the row player's actions are T and B and the column player's actions are L and R.
- Assume that the former uses a mixed strategy $(p, 1-p)$ and the latter a mixed strategy $(q, 1-q)$.
- Then the probabilities for the four possible outcomes are as depicted below

	L	R
T	pq	$p(1-q)$
B	$(1-p), q$	$(1-p)(1-q)$

Mixed strategies

- The utility of the row player is now

$$p[qu_1(T,L)+(1-q)u_1(T,R)]+$$

$$(1-p)[qu_1(B,L)+(1-q)u_1(B,R)]$$

- The magnitudes within square brackets are the utilities associated with pure strategies T and B.
- It is clear that both of them have to be of equal magnitude if p is strictly between zero and unity.
- In other words, the row player has to be indifferent between T and B.
- More generally, in a mixed strategy equilibrium a player has to be indifferent between all pure strategies (actions) in the support of his/her mixed strategy.

Example

Non-standard matching pennies

	H	T
H	$2,-2$	$-1,1$
T	$-3,3$	$2,-2$

Here the row player suggest playing matching pennies, and in order to make the game less monotonic s/he pays 3 to the opponent if s/he chooses tails and the opponent heads, and to even out things s/he pays only 1 if s/he chooses heads and the opponent tails.

Let the row player's mixed strategy be $(p, 1-p)$ and that of the column player's $(q, 1-q)$. Let us determine the row player's best response function.

If the row player chooses H s/he expects $2q - 1(1-q)$, and if s/he chooses T s/he expects $-3q + 2(1-q)$.

It is clear that if q is close to zero then T is the best response, and if q is close to unity then H is the best response.

Mixed strategies

Example

The row player is indifferent between his/her choices if $2q - 1(1-q) = -3q + 2(1-q)$ which is equivalent to $q = \frac{3}{8}$.
Only at exactly this value it is possible that the row player chooses a mixed strategy where $p \in (0,1)$.
In an analogous way one finds that the equilibrium mixed strategy of the row player is $p = \frac{5}{8}$.
The equilibrium pay-off of the row player is $\frac{1}{8}$.
Notice (and confirm by calculation) that using his/her equilibrium strategy the row player earns $\frac{1}{8}$ regardless of the strategy the column player uses.

Example

Notice that a player's mixed strategy is determined by his/her opponent's pay-offs; as a player is indifferent between his/her pure strategies that belong to his/her mixed strategy's support, the only purpose of the mixed strategy is to keep the opponent indifferent, too.

Calculate what happens in the above game to the row player's equilibrium strategy when his/her pay-off for T is slightly raised (say by $\varepsilon > 0$).

One can define weak and strict dominance as before, and the main insight is that players do not use strictly dominated actions in mixed strategy equilibria.

Example

A firm may receive any number of applications for a vacancy from identical workers.
An application consists of a wage demand, and the firm chooses the worker with a lowest wage demand.
On the background there is an economy with many firms and many workers and the workers randomly decide which firm to apply to.
Assume that the number of workers is the same as the number of firms.
Then the number of workers that contact a particular firm is Poisson-distributed with parameter 1.
Thus, a worker that applies to a firm knows that k other workers apply to the same firm with probability $e^{-1}\frac{1}{k!}$, but s/he does not know the exact number of other workers applying to the same firm.
Assume that each worker produces one unit of surplus while employed.

Example

It is clear that there is no pure strategy equilibrium: If each worker's equilibrium wage demand were w then a worker who deviates and demands $w-\varepsilon$ get the job for certain.

For this same reason there is no equilibrium with mass point.

Thus, the equilibrium strategy has to be a mixed strategy.

One can show that the support of the mixed strategy cannot contain gaps so that the mixed strategy F is continuous and has support $[l, L]$.

Let us next construct the mixed strategy.

First, note that $L = 1$; a worker who demands L gets it only if it is the only worker that the firm meets.

The firm is willing to pay up to unity, and thus, any demand less than unity would not be optimal.

Mixed strategies

Example

Demanding $L = 1$ yields utility e^{-1}, i.e. 1 times the probability that there are no other workers applying to the firm.
When a worker makes a demand l s/he gets the job for certain, and as this demand is in the support of the mixed strategy it has to yield the same utility as any other demand in the support.
In particular, it has to yield the same utility as demand $L = 1$.
Thus, $l = e^{-1}$.

Example

Assume that a worker makes a demand $w \in \left(e^{-1}, 1\right)$.
Then his/her utility is given by

$$\sum_{k=0}^{\infty} e^{-1} \frac{1}{k!} (1 - F(w))^k w = e^{-1} e^{(1-F(w))} w$$

This, again, must yield the same utility as other demands in the support of the mixed strategy.

Example

Consequently,

$$e^{-1}e^{(1-F(w))}w = e^{-1}$$

which is equivalent with

$$e^{(1-F(w))}w = 1$$

which is equivalent with

$$1 - F(w) = \ln \frac{1}{w}$$

which is equivalent with

$$F(w) = 1 - \ln \frac{1}{w}$$