Non-co-operative game theory

Lecture 4

November 4, 2015

Bayesian games

- Class of games where players have imperfect information about each others' preferences.
- This means that the players do not know each others' pay-offs.
- There may be both private and common unknown components in the pay-offs.
- Choices are made simultaneously.
- One example is procurement auction.

Example

Entry deterrence.
Entrant thinks of entering a market where there is an incumbent.
Simultaneously they decide whether to enter and whether to expand business.
Profitability of entering depends on whether incumbent expands or not.
Profitability of expanding depends on whether the incumbent's costs are high or low.
The incumbent knows its own costs.
The entrant does not know the incumbent's costs.
The entrant has beliefs about the costs.

Example

The games associated with different costs are as follows

lowcost

	Expand	*Don't*
Enter	$-1, 2$	$1, 1$
Stayout	$0, 4$	$0, 3$

highcost

	Expand	*Don't*
Enter	$-1, -1$	$1, 1$
Stayout	$0, 0$	$0, 3$

Assume that the entrant thinks that costs are low with probability p and high with probability $1 - p$.

Example

One can think of this as a game of three players where Nature is one of them.
Nature moves first and randomises over the type of the incumbent.
For the example, assume that $p = \frac{2}{3}$
Note first that the entrant wants to enter only if the incumbent does not expand.
A high cost incumbent has a dominant strategy Don't.
A low cost incumbent has a dominant strategy Expand.

Example

Entrant's strategy set is $\{Enter, Stayout\}$.
Incumbents strategy set is
$\{f : \{Highcost, Lowcost\} \rightarrow \{Expand, Don't\}\}$.
If entrant enters it gets $\frac{2}{3}(-1) + \frac{1}{3}1 = -\frac{1}{3}$.
If the entrant stays out it gets $\frac{2}{3}0 + \frac{1}{3}0 = 0$.
Nash-equilibrium is then $(Stayout; Expand, Don't)$.

Bayesian games

- The idea is to incorporate imperfect information by modelling players as types.
- The types are possible states of the world.
- Nature chooses the states
- Once the possible types of a players are regarded as players the game is formally similar to the standard normal form game.

Definition

A Bayesian game is $\Gamma = (p, S_i, T_i, u_i)_{i=1}^{N}$ where $\{1, ..., N\}$ is the set of players, S_i is player i's strategy set, p is a probability distribution on $T = T_1 \times ... \times T_N$, T_i is a set of types of player i, and $u_i : S \times T \to \mathbb{R}$ is a Bernoulli utility function of player i, where $S = S_1 \times ... \times S_N$.

Bayesian games

- When $t_i \in T_i$ and (i, t_i) is a player the Nash-equilibrium of the Bayesin game is just like a Nash-equilibrium of a normal form game.

Example

Information may hurt.
There are two states ω_1 and ω_2, and neither player knows the state.
Both players associate belief $\frac{1}{2}$ to both states.
The game is given by

ω_1

	L	M	R
T	$1, 2\varepsilon$	$1, 0$	$1, 3\varepsilon$
B	$2, 2$	$0, 0$	$0, 3$

ω_2

	L	M	R
T	$1, 2\varepsilon$	$1, 3\varepsilon$	$1, 0$
B	$2, 2$	$0, 3$	$0, 0$

Assume that $0 < \varepsilon < \frac{1}{2}$.

Example

Player 2 has unique best response L.
Against T it gives 2ε while M and R give $\frac{3}{2}\varepsilon$
Against B it gives 2 while M and R give $\frac{3}{2}$.
Player 1's unique best response to L is B.
Thus, the unique Nash-equilibrium is (B, L) yielding utility 2 to each player.

Example

Assume that player 2 is informed of the state.
In state ω_1 s/he has a dominant action R.
In state ω_2 s/he has a dominant action M.
From player 1's perspective these actions are chosen with probability $\frac{1}{2}$, each.
Player 1's best response is T.
The Nash-equilibrium is $(T; R, M)$ and players get utilities 1 and 3ε.

Example

Provision of public good.
Two players simultaneously decide whether to provide a public good.
If at least one does so both get utility 1.
Costs of providing the good are private information.

	1	0
1	$1 - c_1, 1 - c_2$	$1 - c_1, 1$
0	$1, 1 - c_2$	$0, 0$

Cost determined independently by a continuous increasing distribution function F on $[\underline{c}, \bar{c}]$.
Assume that $1 \in [\underline{c}, \bar{c}]$
Now there is a continuum of states (or types).

Example

A pure strategy is s_i is a function $s_i : [\underline{c}, \bar{c}] \rightarrow \{0,1\}$.
Player i's pay-off is $u_i(s_i, s_j, c_i) = max\{s_i, s_j\} - c_i s_i$.
A (Bayesian) Nash-equilibrium is $\left(s_i^*, s_j^*\right)$ such that s_i^* maximises
$E_{c_j} u_i\left(s_i, s_j^*(c_j), c_i\right)$.
Player i is interested in the expected probability that player j contributes.
Let $z_j = E_{c_j}\left(s_j^*(c_j) = 1\right)$ be the equilibrium probability that player j contributes.
Player i contributes only if his/her cost is lower than $1 - z_j$.
Thus, $s_i^*(c_i) = 1$ if $c_i < 1 - z_j$ and zero otherwise.

Example

The types of player i who contribute constitute the set $[\underline{c}, c_i^*]$.
Analogously, the types of player j who contribute constitute the set $\left[\underline{c}, c_j^*\right]$.

Now we know that $z_j = F\left(c_j^*\right)$, and the equilibrium cut-off levels must satisfy

$$c_i^* = 1 - F\left(c_j^*\right)$$

$$c_j^* = 1 - F\left(c_i^*\right)$$

Thus, both cut-off levels satisfy $c^* = 1 - F\left(1 - F\left(c^*\right)\right)$.
If there is unique c^*that satisfies the equation then $c_i^* = c_j^* = c^*$.

Example

Two players draw a monetary value from a finite set $Y \subset [0, 1000]$ according to distribution F.
Once they learn the value they write it on a slip of paper.
Then they simultaneously decide whether to exchange the slips (and accordingly the prizes); if both agree the exchange takes place.
This is a Bayesian game with $N = \{1, 2\}$, the type space $T = Y \times Y$, strategy set for each player is $\{exchange, don't\}$, and each player's prior on T is given by $F \times F$.
Player i's utility function is given by $u_i((a, b), y) = y_j$ if $a = b = exchange$ and by $u_i((a, b), y) = y_i$ in all other cases.
Denote the smallest value in Y by $\underline{y}$.

Example

Let $M_i \in Y$ be the highest type of player i who chooses *exchange*.
If $M_i > \underline{y}$ then type $\underline{y}$ of player j finds it optimal to choose *exchange*.
Thus, if $M_i \geq M_j > \underline{y}$ then it is actually optimally for type M_i of player i to choose $don't$.
This is because the expected value of the values that player j exchanges is less than M_i.
Consequently, in any Nash equilibrium only the lowest values are exchanged.