Non-co-operative game theory

Lecture 5

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Games of perfect information in extensive form

Games of (perfect information) in extensive form.

- Timing of moves and sequential structure of the game are important.
- When a player makes a move s/he knows the history of the game.
- Pay-offs are got only when the game ends.
- These games can be represented graphically as game trees.

Example

What difference does the order of moves make?
Two players choose simultaneously actions from the positive real line.
Player i's pay-off is $a_i(c + a_j - a_i)$.
From the FOC one gets the reaction function $a_i = \frac{c+a_j}{2}$.
Symmetric equilibrium is $a_1^* = a_2^* = c$.

Example

Assume now that player 1 moves first, player 2 observes player 1's move and then makes his/her choice.
Player 2 chooses $a_2 = \frac{c+a_1}{2}$.
Player 1 can foresee this and maximises $a_1 \left(c + \frac{c+a_1}{2} - a_1\right)$.
From the FOC one gets $a_1^* = \frac{3}{2}c$ and then $a_2^* = \frac{5}{4}c$.

Definition

Definition. An extensive form game.
An extensive form game consists of
1. A finite set of players N.
2. A set of actions A.
3. A set of histories X where $\emptyset \in X$ and each $\emptyset \neq x \in X$ is $x = (a_1, a_2, ..., a_k)$ for $a_i \in A$, and $(a_1, a_2, ..., a_k) \in X \backslash \emptyset$. Let $A(x)$ be the set of action available to player whose turn it is to move after history x.
4. A probability distribution π on actions $A(\emptyset)$. These are the chance moves (by Nature).
5. A set of end nodes $E = \{x \in X : (x, a) \notin X, a \in A\}$.

Definition

6. A function $i : X \setminus (E \cup \emptyset) \to N$ that tells whose turn it is to move. The set of nodes where it is player i's turn to move is denoted X_i.
7. A partition I of the histories $X \setminus (E \cup \emptyset)$such that if x and x' belong to the same element of the partition then i) $i(x) = i(x')$, and ii) $A(x) = A(x')$. Information set containing x is denoted $I(x)$. Information set where it is player i's turn to move are denoted by I_i.
8. For each player i a von Neumann-Morgenstern utility function $u_i : E \to \mathbb{R}$.

Definition

Definition. Stragegy.
In an extensive form game a strategy is a complete description what to do in each information set.

- In particular a strategy has to tell what to do even after histories that are not reached in equilibrium.
- When each information set contains exactly one history we have a game of perfect information.
- In this the rest of the game at any information set constitutes a new game; a subgame of the original game.
- Any finite extensive form game of perfect information possesses a pure strategy Nash-equilibrium that can be solved by backward induction.
- These equilibria are subgame perfect Nash-equilibria.
- A subgame perfect Nash-equilibrium is such that it induces a Nash-equilibrium in every subgame of the original game.
- This is the major refinement of Nash-equilibrium, and the prediction of outcome in extensive form games of perfect information.

Games of perfect information in extensive form

- $N = \{1, 2\}$, $A = \{L, R, l, r\}$, $X = \{\text{Ø,L,R,Ll,Lr}\}$, $E = \{R, Ll, Lr\}$, $i(\text{Ø}) = 1, i(L) = 2$, $I = \{\{L\}\}$.
- $u_1(R) = 1, u_1(Ll) = 0, u_1(Lr) = 2$, $u_2(R) = 2, u_2(Ll) = 0, u_2(Lr) = 1$.
- There are two Nash-equilibria in the above game: (R, l) and (L, r).
- The first one is not subgame perfect.

Games of perfect information in extensive form

- Centipede game where 2 players can move Down or Continue.
- http://en.wikipedia.org/wiki/Centipede_game
- http://www.econport.org/econport/request?page=man_gametheory
- In the unique subgame perfect Nash-equilibrium they always move Down.

Example

Example. Durable goods monopoly.
Monopoly sells durable good.
Two periods and discounting by factor δ.
Life time valuations of consumers uniform on $[0,1]$.
Measure unity of consumers.
If a consumer with valuation v gets the good in period $t \in \{0,1\}$ at price p s/he gets utility $\delta^t(v-p)$.
Marginal cost of production zero.

Example

Only high valuation consumers buy in SPE in the first period.
Assume that in the second period highest remaining valuation is v_0.
Last period objective is $max_{p_1} p_1 (v_0 - p_1)$.
Optimal choice is $p_1 = \frac{v_0}{2}$.
If the first period price is $p_0$then consumer with valuation v_0 is indifferent between buying and waiting if

$$v_0 - p_0 = \delta \left(v_0 - \frac{v_0}{2} \right)$$

from which

$$v_0 = \frac{2p_0}{2 - \delta}$$

Example

Monopoly's problem is to choose p_0 to maximise

$$p_0\left(1-\frac{2p_0}{2-\delta}\right)+\delta\left(\frac{2p_0}{2-\delta}-\frac{p_0}{2-\delta}\right)\frac{p_0}{2-\delta}$$

Taking the FOC and solving

$$p_0=\frac{(2-\delta)^2}{8-6\delta}$$

In the SPE monopoly set prices $p_0=\frac{(2-\delta)^2}{8-6\delta}$ and $p_1=\frac{2-\delta}{8-6\delta}$ and gets profit $\frac{\left(1-\frac{\delta}{2}\right)^2}{4-3\delta}<\frac{1}{4}$.

Example

Stackelberg competition.
Same setting as in Cournot-competition but firm 1 moves first.
It knows that firm 2 chooses $q_2 = \frac{1-q_1}{2}$.
Incorporating this into its objective firm 1 maximises
$q_1\left(1 - q_1 - \frac{1-q_1}{2}\right)$.
From the FOC one can solve $q_1^* = \frac{1}{2}$ and $q_2^* = \frac{1}{4}$.
Firm 1's action set is $S_1 = \mathbb{R}_+$.
Firm 2's action set is the function space $F = \{f : \mathbb{R}_+ \to \mathbb{R}_+\}$.