# Game theory lecture 4

September 24, 2012

# Finding Nash equilibrium

**Best-response or best-reply functions.**

- We introduced Nash-equilibrium as a profile of actions (an action for each player) such that no player has an incentive to choose a different action (provided that others stick to their choice).

# Finding Nash equilibrium

- It is clear that at a Nash-equilibrium each player's choice is a best response to the other players' choices.
- This may lead one to expect that Nash- equilibrium is a fixed point of the players best-response functions (or correspondences to be precise).
- The best-response function is dened as follows

**Definition.** In a normal form game $\Gamma = \left(N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N}\right)$player $i$'s best-response function is defined as $B_i(a_i, a_{-i}) = \{a_i \in A_i : u_i(a_i, a_{-i}) \geq u_i(a_i', a_{-i})\}$ for all $a_i' \in A_i$.

# Finding Nash equilibrium

- Notice that unlike in Osborne I have defined the best-response function such that the argument includes all the players' choices; this is convenient in some instances but it is of no importance.
- Notice also that even though I, and Osborne, call it a function it is not; the best-response may contain many elements, and typically such objects are called correspondences.

**Definition.** In a normal form game $\Gamma = \left(N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$ an action profile $a* \in \times_{i\in N} A_i$ is a Nash equilibrium if $a_i^* \in B_i(a*)$ for all $i \in N$.

**Example1.** Cournot competition

- Consider a standard linear inverse demand $p = 1 - q$.
- There are two firms and each chooses how much to offer for sale simultaneously.
- Assume that marginal costs of production are zero.
- Assume that firm 2 produces $q_2$.
- Firm 1's best- response is given by $q_1(q_2) = argmax_{q_1}(1 - q_1 - q_2)q_1$.
- This can be found by taking the first order condition $q_1 = \frac{1-q_2}{2}$.
- As the situation is symmetric firm 2's best-response is evidently given by $q_2 = \frac{1-q_1}{2}$.

# Finding Nash equilibrium

- Solving the pair of equations yields the symmetric Nash equilibrium $\left(\frac{1}{3}, \frac{1}{3}\right)$.
- This, however, is not the only Nash-equilibrium of the model.
- The other equilibria are not found by straightforward use of calculus but one has to think about best-response functions.
- All action profiles where $q_1 \in [1, \infty]$ and $q_2 \in [1, \infty]$ are also Nash-equilibria.
- Try to figure out what is their relation to dominance!

# Finding Nash equilibrium

**Example2.**

- A common resource is used by $n$ firms.
- Firm $i$'s production is given by $x_i(1-(x_1+x_2+\ldots+x_n))$ as long as $x_1+x_2+\ldots+x_n<1$ and zero otherwise.
- If firms maximise production the best-response function of firm $i$ is found by determining the first order condition

$$1-(x_1+x_2+\ldots+x_n)-x_i=0$$

Since the situation is symmetric it is natural to look for a symmetric Nash equilibrium where all firms use the same strategy $x$.

- The FOC becomes then

$$1-(n+1)x=0$$

and the Nash equlibrium is given by $\left(\frac{1}{n+1},\ldots,\frac{1}{n+1}\right)$.

- Notice again that there are other equilibria: If all the firms i choose $x_i=1$then eveyone gets zero and no firm can improve

# Finding Nash equilibrium

**Example3.**

- Bertrand competition is like Cournot competition except that the firms choose prices instead of quantities.
- Consumers buy from the firm with the lowest price; if prices are equal the firms divide the market.
- Best-response of firm $i$ is given by

$$B_i(p_i, p_j) = \begin{cases} \{p_i : p_i > p_j\} \text{ if } p_j < 0 \\ \{p_i : p_i \geq 0\} \text{ if } p_j = 0 \\ \emptyset \text{if } 0 < p_j \leq p^m \\ \{p^m\} \text{ if } p^m < p_j \end{cases}$$

  where $p^m$ denotes the monopoly price.
- Graphing the best-response functions it is immediate that the unique Nash equilibrium is $(0, 0)$.

**Example4.**

- Let us consider so called second-price sealed-bid auction.
- There are $n \geq 2$ bidders, and an indivisible object for sale.
- Player $i$ has valuation $v_i$ for the object, and if s/he gets it at price $p$ his/her utility is $v_i - p$.
- Players' action sets are positive real numbers from which they choose their bids $b_i$.
- The rules are such that the highest bidder wins, and pays the second highest bid (if there are draws some known rule is applied).
- Change the names of the bidders so that the order of the valuations is $v_1 > v_2 > ... > v_n$.

# Finding Nash equilibrium

- This game is remarkable in that it has a Nash-equilibrium in dominant strategies.
- Bidding one's own valuation is a dominant strategy: Changing the bid does not affect the price conditional that the player would win anyway.
- If a player bids less than his/her valuation s/he reduces his/her chances of getting the object as s/he might bid so low that his/her bid is not anymore the highest.
- Conditional on winning nothing happens to price as s/he still has to pay the second highest bid.
- If a player bids more than his/her valuation nothing happens if his/her bid was the highest to start with.
- But if it was not s/he might win the object but then s/he has to pay more than his/her valuation.
- This is a situation where revealing ones true preferences is a dominant strategy equilibrium.

# Finding Nash equilibrium

- There are other equilibria.
- If $n = 3$, for instance, and the valuations are $v_1 = 10$, $v_2 = 6$ and $v_3 = 2$ the following is a Nash-equilibrium: $(b_1 = 3, b_2 = 97, b_3 = 5)$.

**Example5.**

- All-pay auction is a game where the highest bidder wins and all bidders pay their bid.
- It can be used to model political lobbying, or rent seeking, where $n$ players invest in, say, bribing a politician who has a right to grant a monopoly or some prize.
- Let us assume that the probability of winning the price for player $i$ is given by $p_i = \frac{b_i}{\sum_{j=1}^{n} b_j}$ where $b_i$ is the bribe by player $i$.
- If the value of the monopoly right is $V$ then player $i$'s objective is $max_{b_i} p_i V - b_i$.

# Finding Nash equilibrium

- Again we determine the first-order condition

$$\frac{\sum_{j=1}^{n} b_j - b_i}{\left(\sum_{j=1}^{n} b_j\right)^2} V - 1 = 0$$

- Then we focus on a symmetric equilibrium which means that $b_i = b_j = b$ for all $j \in \{1, 2, ..., n\}$.
- Inserting this information to the FOC we can solve for the symmetric Nash equilibrium $b^N = \left(\frac{n-1}{n^2} V, ..., \frac{n-1}{n^2} V\right)$.
- The total expenditure, pure waste if the politician's utility is ignored, is $nb = \frac{n-1}{n} V$; if there are many bidder or lobbyists almost all of the value is wasted in the rent seeking activity.

# Finding Nash equilibrium

**Example6.**

- Two players have to divide a cake.
- Both state simultaneously what is the share they want.
- If the shares sum to at most unity the cake is divided accordingly.
- If the shares exceed unity neither player gets anything.
- Any $(x, 1-x)$, $x \in [0,1]$ where $x$ is the share of player 1 constitutes a Nash-equilibrium.
- There are others!

# Mixed strategies

- In many games there are no Nash-equilibria in pure strategies.
- An example is Matching Pennies game below where the players simultaneously choose Heads or Tails, and if the choices are the same player 1 wins one unit from player 2, and if they differ player 2 wins one unit from player 1.

| | $H$ | $T$ |
|---|---|---|
| $H$ | $1,-1$ | $-1,1$ |
| $T$ | $-1,1$ | $1,-1$ |

# Mixed strategies

- The solution to this problem involves extending the action spaces of the players to include probability distributions.
- When this extension is made I shall call the players' action spaces strategy spaces.
- Instead of choosing single actions the players are allowed to choose probability distributions over the original actions.
- The probability distributions are then called mixed strategies.
- In the above example players would choose probability distributions $(p, 1-p)$ where $p$ is the probability of choosing action H and $1-p$ is the probability of choosing action T.

# Mixed strategies

- Considering mixed strategies as objects of choice presents some problems.
- If a player expects his/her opponent to use a mixed strategy how should s/he evaluate the utility that a particular action gives him/her?
- From the player's point of view s/he is participating in a lottery/gamble, and the situation is like that of a decision maker under uncertainty.
- We know that decision making under uncertainty can be handled with relative ease if the decision maker has von Neumann-Morgenstern preferences, i.e., if his/her preferences have a utility representation in the expected utility form.

# Mixed strategies

- To remind, if a decision maker has von Neumann-Morgenstern type preferences s/he evaluates the expected utility of a lottery $q$ on a numerable set $A$ by

$$U(q) = \sum_{a \in A} q(a)u(a)$$

  where $u$ is many times called the Bernoulli utility function while $U$ is the von Neumann-Morgenstern utility function.
- If the set $A$ is not numerable the sum must be replaced by the proper integral.
- Since we have assumed all the time that the players have von Neumann-Morgenstern utilities no problems should arise.
- It is important to keep in mind that all the pay-offs are in von Neumann- Morgenstern utility units, and for instance issues of risk do not arise as the numbers already reflect these matters.

**Definition.** Mixed extension
The mixed extension of a normal form game $\Gamma = \left(N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$ is a normal form game where player $i$'s action set is replaced by $S_i = \left\{p : \int_{A_i} dp = 1\right\}$.

**Definition.** Nash equilibrium Consider a normal form game $\Gamma = \left(N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$, and its mixed extension $\Gamma^{me} = \left(N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}\right)$. A Nash equilibrium is a vector of strategies $s = (s_1, s_2 ..., s_n)$ such that for all players $i \in N$ $U_i(s_i, s_{-i}) \geq U_i(s_i', s_{-i})$ for all $s_i' \in S_i$.

# Mixed strategies

- Notice that in the denition the Nash-equilibrium applies both the to normal form game and its mixed extension.
- In the sequel we do not make any difference between the two, and when one is looking for Nash-equilibria it is understood that one is looking for equilibria both in pure and mixed strategies.
- The mixed extension makes it possible to show Nash-equilibrium existence in a large class of games.

# Mixed strategies

**Theorem.** Existence in finite games Every finite normal form game has a mixed strategy equilibrium.

# Finding mixed strategy Nash equilibrium

- To find mixed Nash-equilibria in simple games it is useful to consider a 2x2-game where the row player's actions are $T$ and $B$ and the column player's actions are $L$ and $R$.
- Assume that the former uses a mixed strategy $(p, 1-p)$ and the latter a mixed strategy $(q, 1-q)$.
- Then the probabilities for the four possible outcomes are as depicted below

| | $L$ | $R$ |
|---|---|---|
| $T$ | $pq$ | $p(1-q)$ |
| $B$ | $(1-p)q$ | $(1-p)(1-q)$ |

# Finding mixed strategy Nash equilibrium

- The utility of the row player is now

$$p\left[qu_1(T,L)+(1-q)u_1(T,R)\right]+$$

$$(1-p)\left[qu_1(B,L)+(1-q)u_1(B,R)\right]$$

- Square brackets contain the utilities associated with pure strategies $T$ and $B$.
- Both of them have to be of equal magnitude if $p$ is strictly between zero and unity.
- In other words, the row player has to be indifferent between $T$ and $B$.
- More generally, in a mixed strategy equilibrium a player has to be indifferent between all pure strategies (actions) in the support of his/her mixed strategy.

# Finding mixed strategy Nash equilibrium

**Example.** Non-standard matching pennies

| | $H$ | $T$ |
|---|---|---|
| $H$ | $2, -2$ | $-1, 1$ |
| $T$ | $-3, 3$ | $2, -2$ |

- Here the row player suggests playing matching pennies, and in order to make the game less monotonic s/he pays 3 to the opponent if s/he chooses tails and the opponent heads.
- To even out things s/he pays only 1 if s/he chooses heads and the opponent tails.
- Let the row player's mixed strategy be $(p, 1-p)$ and that of the column player $(q, 1-q)$.
- The row player has to be indifferent between his/her choices.

# Finding mixed strategy Nash equilibrium

- If s/he chooses $H$ s/he expects

$$2q - 1(1-q)$$

- If s/he chooses $T$ s/he expects

$$-3q + 2(1-q)$$

- These have to be equal or

$$2q - 1(1-q) = -3q + 2(1-q)$$

- The solution to this equation is $q = \frac{3}{8}$.

# Finding mixed strategy Nash equilibrium

- Analogously one finds the other mixed strategy $p = \frac{5}{8}$. DO THIS.
- The equilibrium pay-off of the row player is $\frac{1}{8}$.
- Using this strategy the row player can guarantee pay-off $\frac{1}{8}$ regardless of what the column player does. CONFIRM THIS.

# Finding mixed strategy Nash equilibrium

- Notice that a player's mixed strategy is determined by his/her opponents pay- offs.
- As a player is indifferent between his/her pure strategies that belong to his/her mixed strategy's support, the only purpose of the mixed strategy is to keep the opponent indifferent.
- Calculate what happens in the above game to the row player's equilibrium strategy when his/her pay-off for $T$ is slightly raised (say by $\varepsilon > 0$).

**Example.** Voting

- There are two candidates $A$ and $B$ who are supported by $n_A$ and $n_B$ voters where $n_A > n_B$.
- The candidate who gets more votes is elected.
- If a voter's favourite is elected s/he receives utility 1, and zero otherwise.
- Voting is costly and we denote the cost by $c > 0$.
- Denote the equilibrium probabilities of voting by $a$ and $b$.
- Let us study the supporters of candidate $A$.

# Finding mixed strategy Nash equilibrium

- If a supporter does not vote his/her utility is

$$Pr(A\, wins) \cdot 1 =$$

$$\sum_{i=1}^{n_A-1} \binom{n_A-1}{i} a^i (1-a)^{n_A-1-i} \sum_{j=1}^{min\{i-1,n_B\}} \binom{n_B}{j} b^j (1-b)^{n_B-j} +$$

$$\frac{1}{2} \sum_{i=0}^{n_B} \binom{n_A-1}{i} a^i (1-a)^{n_A-1-i} \binom{n_B}{i} b^i (1-b)^{n_B-i}$$

# Finding mixed strategy Nash equilibrium

- If a supporter votes his/her utility is

$$Pr(A\,wins)\cdot 1 - c =$$

$$\sum_{i=0}^{n_A-1}\binom{n_A}{i}a^i(1-a)^{n_A-1-i}\sum_{j=0}^{min\{i,n_B\}}\binom{n_B}{j}b^j(1-b)^{n_B-j}+$$

$$\frac{1}{2}\sum_{i=0}^{n_B}\binom{n_A-1}{i}a^i(1-a)^{n_A-1-i}\binom{n_B}{i+1}b^{i+1}(1-b)^{n_B-i-1}-c$$

# Finding mixed strategy Nash equilibrium

- These two expressions have to be equal in a Nash equilibrium in order to make the voter indifferent between voting and not voting.
- One gets an analogous equations for a supporter of $B$, and from these one can determine a symmetric equilibrium.
- Typically there are multiple symmetric equilibria.
- Let us consider a simple case where $n_A = 2$ and $n_B = 1$.

# Finding mixed strategy Nash equilibrium

- The equations look as follows.
- If a supporter of $A$ does not vote $A$ wins with probability

$$\frac{1}{2}(1-a)(1-b)+\frac{1}{2}ab+a(1-b)$$

- If s/he votes $A$ wins with probability

$$\frac{1}{2}(1-a)b+(1-b)$$

- In a mixed strategy equilibrium the voter has to be indifferent between the choices or

$$\frac{1}{2}(1-a)(1-b)+\frac{1}{2}ab+a(1-b)=\frac{1}{2}(1-a)b+(1-b)-c$$

- From this one can solve

$$b=\frac{1-a-2c}{a}$$

# Finding mixed strategy Nash equilibrium

- If the supporter of $B$ does not vote $B$ wins with probability

$$\frac{1}{2}(1-a)^2$$

- If s/he votes $B$ wins with probability

$$\frac{1}{2}2a(1-a)+(1-a)^2$$

- In equilibrium

$$\frac{1}{2}(1-a)^2=\frac{1}{2}2a(1-a)+(1-a)^2-c$$

- One can solve

$$a=\sqrt{1-2c}$$

- Are the any values of $c=0,15$ such that all agents vote with positive probability? DO THIS

## Exercises

1. There is a vacancy available. There are two potential job applicants $A$ and $B$. There are three equally likely states of the world $s_A$, $s_B$ and $s_{AB}$. In the first state only applicant $A$ is interested in the job, in the second state only applicant $B$ is interested in the job and in the last state both of them are interested in the job.
A job applicant must make a wage demand to an employer. The maximum the employer is willing to pay is 1. A job applicant only knows whether s/he is interested in the job but s/he does not know the exact state of the world. Determine the applicants' equilibrium strategy.

2. Determine all the equilibria of the following game

| | $c$ | $d$ |
|---|---|---|
| $a$ | $5,2$ | $2,2$ |
| $b$ | $1,1$ | $3,6$ |

3. Determine the pure strategy Nash equilibria of the following game

| | ö | ä | z | x | y | w | v | u |
|---|---|---|---|---|---|---|---|---|
| a | 23, 4 | 19, 300 | 9, 17 | 20, 30 | 5, 4 | 59, 2 | 49, 5 | 1, 2 |
| b | 5, 8 | 39, 11 | 0, 6 | 4, 6 | 48, 99 | 78, 666 | 6, 7 | 5, 4 |
| c | 9, 55 | 10, 8 | 0, 55 | 7, 7 | 33, 55 | 66, 7 | 90, 90 | 44, 90 |
| d | 6, 4 | 1, 3 | 6, 0 | 0, 2 | 23, 3 | 11, 3 | 5, 5 | 6, 44 |
| e | −4, 7 | 64, 90 | 33, 5 | 58, 0 | 10, 19 | 9, 11 | 9, 3 | 5, 7 |
| f | 88, 5 | 3, 55 | 100, 78 | 0, 0 | 77, 5 | 7, 5 | 5, 78 | 7, 23 |
| g | 4, 66 | 6, 5 | 6, 45 | 4, 66 | 9, 77 | 10, 0 | 20, 5 | 9, 5 |
| h | 77, 4 | 45, 66 | 7, 8 | 9, 4 | 3, 4 | 33, 34 | 0, 5 | 5, 0 |