Game theory lecture 6

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- This is a class of games where players have inperfect information about each other's preferences, and where choices are made simultaneously.
- Another (informal) interpretation is that the players are not certain about which game they are playing.
- The next example presents an entry deterrence game between an incumbent and an entrant.

Example1. Entry deterrence

	expand		don't	
enter	-1	2	1	1
stayout	0	4	0	3

State s_L

	expand		don't	
enter	-1	-1	1	1
stayout	0	0	0	3

State s_H

- Assume that an entrant ponders upon entering a market where there is an incumbent.
- The entrant and the incumbent decide simultaneously whether to expand the business (incumbent) and whether to enter or not (entrant).
- The profitability of entering depends on whether the incumbent expands of not, and the profitability of expanding depends on whether the incumbent's costs are low or high.
- The incumbent knows whether its costs are low or high.
- The entrant, though, does not know this.
- To be able to analyse the situation the entrant (as well as we) must have a belief about the likelihood of the incumbent's costs.

- Assuming that the entrant knows (or has beliefs) how the incumbent plays depending on its costs this is an uncertain decision situation from the entrant's point of view.
- Like in the expected utility framework we can think that there are two states of the world s_L and s_H .
- Assume that the entrant attaches probabilities p and $1-p$ to these states.

- One way of modelling the situation (this is different from Osborne) is that there are three players Nature, the incumbent and the entrant.
- The incumbent can be of two types: Low cost or high cost.
- Nature chooses the type with probabilities known to the players.
- This is called the prior belief.
- For this example, let us assume that the probability of the incumbent being high type is $1/3$.

Bayesian games

- To proceed note that the entrant only wants to enter if the incumbent does not expand.
- A high cost incumbent has a dominant strategy of not expanding, and a low cost incumbent has a dominant strategy of expanding.
- The entrant's strategy is simply either to enter or to stay out, while the incumbent's strategy is a function from its set of types to the set $\{Expand, Don't\}$.
- If the entrant chooses to enter it expects to get $\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (-1) = -\frac{1}{3}$.
- If the entrant stays out it expects to get $\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot 0 = 0$.

- In equilibrium the incumbent chooses its dominant action given its type, and the entrant chooses to stay out.
- So, a Nash-equilibrium is $(Stayout; Expand, Don't)$ where the first choice of the incumbent refers to the case where the state is s_L, and the second choice to the case where the state is s_H.

Bayesian games

- The basic idea in incorporating imperfect information into the games of above type is to model the players as types, and possible types as the states of the world.
- In Osborne the player 'Nature' is replaced by a signal-technology; each player gets a signal of the state of nature, and typically the signal tells a receiving player's type accurately but provides imperfect information about the other players' types.
- Once the possible types of each player are regarded as players, the game is formally similar to a standard normal form game with some uncertainty as to other players' types, and the machinery of normal form games is available.

Definition. A Bayesian game
A Bayesian game $\Gamma = \left(N, (S_i)_{i \in N}, \Omega, (p_i)_{i \in N}, (\tau_i)_{i \in N}, (u_i)_{i \in N}\right)$ consists of the set of players N, set of strategies for each player S_i, the set of states Ω, a signal function for each player $\tau_i : \Omega \to T_i$ and a utility function for each player $u_i : \times_{i \in N} S_i \times \Omega \to \mathbb{R}$.

- When player i receives signal t_i s/he is referred to as type t_i.
- One way of presenting the previous game as a Bayesian game is as follows:
- $N = \{Incumbent, Entrant\}$, $S_I = \{Expand, Don't\}$ and $S_E = \{Enter, Stayout\}$, $\Omega = \{Low, High\}$, $p_I = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $p_E = \left(\frac{2}{3}, \frac{1}{3}\right)$, $\tau_I(Low) = 21$, $\tau_I(High) = 121$ and $\tau_E(Low) = \tau_E(High) = 79$.
- In addition $u_I(Enter, Expand, Low) = 2$ and so on.

Bayesian games

- To a Bayesian game corresponds a normal form game where the players are all the types with corresponding strategy set, and where player (i, t_i) has Bernoulli utility function u_i.
- Nash equilibrium of a Bayesian game is defined to be Nash equilibrium of the corresponding normal form game.
- Bayesian games exhibit, at the first thought, some counter intuitive properties as the next example from the text book demonstrates.

Example2. Information may hurt

- There are two states ω_1 and ω_2 and neither player knows the state.
- Each player's prior belief is that the states are equally likely.
- The game is depicted below

ω_1	L	M	R
T	$1,2\varepsilon$	$1,0$	$1,3\varepsilon$
B	$2,2$	$0,0$	$0,3$

ω_2	L	M	R
T	$1,2\varepsilon$	$1,3\varepsilon$	$1,0$
B	$2,2$	$0,3$	$0,0$

Bayesian games

- First we assume that $0 < \varepsilon < \frac{1}{2}$.
- Player2 has a unique best response L; against T it yields 2ε while M and R yield $\frac{3}{2}\varepsilon$.
- Against B it yields 2 while M and R yield $\frac{3}{2}$.
- Player 1's unique best response against L is B in both states.
- The unique Nash-equilibrium is (B, L) and both players get pay-off 2.

- Assume then that player 2 is informed about the state.
- But now player 2 has in both states a dominant strategy M or R.
- From player 1's perspective actions M and R are chosen with equal probabilities.
- His/Her optimal response is to choose T.
- Player 1's pay-off is then 1 and player 2's pay-off is 3ε.

Example3. Osborne page 284

- There are two players and three states $\Omega = \{\alpha, \beta, \gamma\}$.
- The players' information can be depicted using information partitions.
- Player 1: $\{\{\alpha\}, \{\beta, \gamma\}\}$.
- Player 2: $\{\{\alpha, \beta\}\{\gamma\}\}$.
- When player 1 knows that the state is in $\{\beta, \gamma\}$ his/her belief is that the true state is β is $\frac{3}{4}$.
- When player 2 knows that the state is in $\{\alpha, \beta\}$ his/her belief is that the true state is β is $\frac{1}{4}$.

Bayesian games

α	L	R
L	$2,2$	$0,0$
R	$3,0$	$1,1$

β	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

γ	L	R
L	$2,2$	$0,0$
R	$0,0$	$1,1$

Bayesian games

- The unique Nash-equilibrium of this game is (R, R).
- If player 1 knows that the state is α to choose then s/he has a dominant action R.
- If player 2 knows that the state is in $\{\alpha, \beta\}$ action L gives at best $\frac{3}{4}0 + \frac{1}{4}2$, while the minimum from action R is $\frac{3}{4}1 + \frac{1}{4}0$.
- Thus player 2 chooses R.
- If player 1 knows that the state is in $\{\beta, \gamma\}$ action L gives at best $\frac{3}{4}0 + \frac{1}{4}2$, while the minimum from action R is $\frac{3}{4}1 + \frac{1}{4}0$.
- Thus player 1 chooses R.
- If player 2 knows that the state is γ action L gives zero while R gives 1.

Example4. Provision of public good

- Two players simultaneously decide whether to provide a public good.
- If at least one of them provides it both get utility 1.
- The players have (potentially) different costs which are their private information.
- The pay-offs are shown below where 1 signifies contributing and 0 signifies not contributing.

	1	0
1	$1-c_1, 1-c_2$	$1-c_1, 1$
0	$1, 1-c_2$	$0, 0$

Bayesian games

- Assume that the players' costs are independent draws from a continuous and strictly increasing distribution F on $[\underline{c}, \bar{c}] \ni 1$.
- Notice that there is a continuum of states.
- In this game a pure strategy s_i is a function $s_i : [\underline{c}, \bar{c}] \rightarrow \{0, 1\}$.
- Player i's pay-off is

$$u_i(s_i, s_j, c_i) = max\{s_i, s_j\} - c_i s_i$$

- A Bayesian Nash equilibrium is a pair of strategies (s_1^*, s_2^*) such that for each player i and each cost c_i, s_i^* maximises

$$E_{c_j} u_i \left(s_i, s_j^*(c_j), c_i \right)$$

- If one is industrious enough to write this explicitly one notices that the important thing from player i's point of view is the probability that player j contributes.
- Let $z_j = E\left(s_j^*(c_j) = 1 \right)$ be the expected equilibrium probability that j contributes.
- It is clear that i contributes only if his/her cost is less than $1 - z_j$.

Games in extensive form

- Consequently, $s_i^*(c_i) = 1$ if and only if $c_i \leq 1 - z_j$ (zero otherwise).
- This means that the types of player i (or the states) who contribute are in interval $[\underline{c}, c_i^*]$.
- By analogous reasoning the types of player j who contribute are in an interval $\left[\underline{c}, c_j^*\right]$.

- Now that we know $z_j = F\left(c_j^*\right)$ and that the equilibrium cut-off levels must satisfy

$$c_i^* = 1 - F\left(c_j^*\right)$$

$$c_j^* = 1 - F\left(c_i^*\right)$$

Exercise1. In the last example let F be the uniform distribution on $[0,2]$. Show that in the symmetric equilibrium $c_1^* = c_2^* = \frac{2}{3}$.

Exercise2.

a) A buyer has valuation one or zero with equal probabilities. A seller makes an offer to the buyer, and the buyer accepts or rejects. Draw the game tree and determine the equilibrium when there is perfect information.

b) Do the same thing when the buyer's information is private.

c) Do the same as in b) when the buyer makes the offer.