Game theory lecture 1

September 9, 2013

Goals

1. Ability to formulate socially interesting and relevant situations of strategic interaction as games.
2. Ability to solve simple games mentioned above.

Structure by lectures

1. Some games from recreational mathematics.
2. More games from recreational mathematics.
3. Basic concepts and examples of games from economics and social sciences; normal form games.
4. Equilibria in mixed strategies.
5. Extensive form games.
6. Bayesian games.
7. Bargaining.
8. Economic applications and exercises.

Some literature

1. There are no literature requirements for this course on top of the lectures; the slides are available on my homepage with some delay.
2. Osborne Martin: "An introduction to game theory" is useful but a little too advanced for this course.
3. Lambertini Luca: "Game theory in the social sciences. A reader-friendly guide" is at parts good and on the suitable level. Parts of the book are simply incorrect or astonishing speculation.
4. The following site on the net contain exercises, texts and experiments and is highly recommended: http://www.gametheory.net/
5. See which events practitioners of game theory regard as interesting/significant http://www.econ.canterbury.ac.nz/personal_pages/paul_walker/gt/

Recreational games

- One can formulate the following examples as games if necessary but most of the time it is quite trivial as winning the game can be normalised to be worth unity and losing it worth zero.
- Let us present a series of examples.

Example1. A row of plus-signs, say $n > 0$ of them, is on the board

$$+++++++++++++$$

Two players remove them in turns; player1 starts. A player can remove one sign or two adjacent signs. The player who removes the last sign is the winner. Does either player have a winning strategy?

Example2. Three players are in a room. A red hat or a blue hat is randomly put on each player. A player cannot see the colour of his/her own hat but s/he can see the other players' hats. The players are not allowed to communicate, and at a predetermined time the players have to simultaneously try to guess the colour of their own hats.

The possible actions for the players are $A = \{red, blue, noguess\}$so that they can also refrain from guessing. The rules of the game are that the players win as a team if at least one guess is correct and no guess is incorrect. In all other cases the players lose. Before going to the room the players can meet and agree on a strategy. Which strategy maximises their chances of winning?

Example3. $2n+1$ cards numbered by interegers from $-n$ to n are placed face up in a row. Two players in alternating turns take a card, and the first one to gather three cards whose sum is zero wins. If there are no more cards left and neither player has three cards that sum to zero it is a draw. Is there a winning strategy?

Let $n \geq 5$, try to see yourself what happens if $n < 5$. Player1 chooses 0 . Assume without loss of generality that player2 chooses a negative number $-a$.
One can divide the possibilities into the following four cases.

i) Assume that a$\notin \{1, n-1, n\}$. Then player1 chooses 1. Now player2 must -1 . Player1 must choose $a+1$, but s/he also wins as player2 cannot defend against both $a+1$ and $a+2$.
ii) Assume that $-a = -1$. Player1 chooses 2 . Player2 must choose -2 . Player1 must choose 3, but s/he also wins as player2 cannot defend against both -3 and -4.

iii) Assume that $-a = -(n-1)$. Player1 chooses 1 . Player2 must choose -1 . Player1 must choose n, and then player2 must choose $-n$. This does not create a possible winning position and player1 can choose 2 which leads to victory.
iv) Assume that $-a = -n$. Player1 chooses 1 . Player2 must choose -1 . This does not create a possible winning position and player1 can choose 2 which leads to victory.

Example4. In this game either all the players win or none of them wins. There are one hundred players whose names are $1, 2, \ldots, 100$. One at a time they enter a room where there are 100 closed doors, and behind each door is a slip of paper with a number between 1 and 100 inclusive. The players do not know which number is behind which door but they know that any of the numbers is behind some door. Each player is allowed to open 50 doors after which s/he closes the doors and exits the room so that s/he does not meet the remaining players. The other players cannot see which doors are opened, and consequently there is no communication between the players who have opened the doors and who have not opened the doors. If all the players find their name they get to live, otherwise they are killed. Determine a strategy that maximises their chances of survival.

Solution.

- We do not show optimality and we start with a simpler problem of 10 prisoners.
- Notice that if each player opens 5 lockers randomly each succeeds with probability $\frac{1}{2}$.
- The probability that all of them find their names is $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$.
- Consider strategy such that Player with name $i \in \{1, 2, \ldots 10\}$ opens door number i.
- If s/he finds number i s/he is done.
- If s/he finds number $j \neq i$ then s/he opens door number j.
- And s/he keeps going like this until s/he finds his/her name, or s/he has opened 5 doors.

- S/he does not find his/her name if it is in a cycle that is longer than 5.
- So let us count the number of permutations which have a cycle of length six or longer.
- Choose any six numbers.
- This can be done in $\binom{10}{6}$ ways.
- There are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways to arrange the six numbers in a cyclic order.
- The remaining 4 number can be in whatever order and there are 4! of them.

- So the probability that there is a 6-cycle is $\frac{\binom{10}{6}5!4!}{10!} = \frac{1}{6}$.
- Calculations are analogous for longer cycles, and the probability that there are no cycles longer than 5 is given by

$$1 - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} \approx 0,354$$

- Completely analogous calculation for 100 prisoners yields probability of winning $0,312$.

- If the number of players is increased, and denoted by $2n$, we get the winning probability

$$1-\sum_{k=1}^{n}\frac{1}{n+k}$$

- Think of $\sum_{k=1}^{n}\frac{1}{n+k}$ as an upper Riemann sum for $\int_n^{2n}\frac{1}{x+1}dx$ and as a lower Rieman sum for $\int_n^{2n}\frac{1}{x}dx$.
- We get bounds

$$\int_n^{2n}\frac{1}{x+1}dx = ln\left(2-\frac{1}{n+1}\right) \leq \sum_{k=1}^{n}\frac{1}{n+k} \leq \int_n^{2n}\frac{1}{x} = ln2$$

- Consequently, when the number of prisoners grows without limit, or $n \longrightarrow \infty$, the probability of survival goes to $1-ln2 \approx 0,307$.