# Game theory lecture 2

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Example1. Game of NIM

There are several heaps of sticks and two players who move alternately. An allowed move is to remove any number of sticks from any heap but from only one heap at a time. The player who removes the last stick is the winner. Since both the number of heaps and sticks in any heap are finite the game ends after a finite number of moves. Code the state of the game by $n^k = (n_1, n_2, ..., n_k)$ meaning that there are $k$ heaps with $n_i$ sticks in heap $i$; let us call this a position and denote the set of possible positions by $\Pi = \cup_{h\in\mathbb{N}}\mathbb{Z}_+^h$.

Let us adopt some notation:
$N = \{n \in \Pi | nextplayer\ can\, ascertain\, victory\, when\, the\, position\, is\, n\}$ and $P = \{n \in \Pi | all\ moves\, lead\, to\, N\, when\, the\, position\, is\, n\}$. Since this is a finite game (progressively bounded impartial game) $N$ and $P$ partition all the NIM-games, or there is always a player with a winning strategy.

- It is clear that $(1,0)$, $(0,1)$, $(n_1,0)$ and $(0,n_2)$ belong to $N$, while $(1,1)$ belongs to $P$.
- Also $(1,2)$ and $(2,1)$ belong to $N$.
- Position $(n,n)$ belongs to $P$ and if $m \neq n$ position $(m,n)$ belongs to $N$.
- One can say something about positions with three and four heaps but things start getting complicated.
- It is better to try to come up with some general results.

Lemma. If $n^k$ and $n^l$ are positions a concatenated position is denoted by $\left(n^k, n^l\right)$.

a) If $n^k$ and $n^l$ belong to $P$ then $\left(n^k, n^l\right)$ belongs to $P$.

b) If $n^k$ belongs to $P$ and $n^l$ belongs to $N$ then $\left(n^k, n^l\right)$ belongs to $N$.

c) If $n^k$ and $n^l$ belong to $N$ then $\left(n^k, n^l\right)$ may belong to $N$ or to $P$.

*Proof.* Notice first that trivially if $(n^k, n^l)$ has at most one stick then either $n^k$ and $n^l$ both have zero sticks and $(n^k, n^l)$ belongs to $P$, or $n^k = 1$ and $n^l = 0$ (or vice versa) in which case the former belongs to $N$ and the latter to $P$, and $(n^k, n^l)$ to $N$ . Proceed by induction and assume that whenever $(n^k, n^l)$ has at most $n$ sticks it holds that if $n^k$ and $n^l$ belong to $P$ then $(n^k, n^l)$ belongs to $P$, and if $n^k$ belongs to $P$ and $n^l$ belongs to $N$ then $(n^k, n^l)$ belongs to $N$.

Proof (Continued). Assume next that $(n^k, n^l)$ has at most $n+1$ sticks. If $n^k$ belongs to $P$ and $n^l$ belongs to $N$ then the next player can make $n^l$ such that the new position belongs to $P$ which creates a concatenation of two $P$-positions. This new position has at most $n$ sticks and by the induction hypothesis it must belong to $P$. But then it must be the case that $(n^k, n^l)$ belongs to $N$.

Proof (Continued) If $n^k$ and $n^l$ belong to $P$ then the next player must take sticks from one of the piles, let them belong to $n^l$. But this always turns $n^l$ a position that belongs to $N$, and the new position is a concatenation of a position that belongs to $P$ and a position that belongs $N$. It also has at most $n$ sticks, and by the induction hypothesis belongs to $N$. Thus, the original position belongs to $P$.

To conclude the proof note that a single pile is in $N$, while $(1,1)$ belongs to $P$, and $(1,2)$ belongs to $N$. QED

**Definition.** The Nim-sum of $m, n \in \mathbb{N}$ is got as follows. Express $n$ and $m$ in binary form (radix), and sum the digits in each column modulo 2. The resulting binary number is the Nim-sum of $n$ and $m$. Denote it by $n \oplus m$.

This can be generalised in an obvious way to any finite number of numbers.

Example. The Nim-sum of 3, 9 and 13: $3 = 1 \times 2^1 + 1 \times 2^0 = 11_b$, $9 = 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 1001_b$ and $13 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 1101_b$. Now summing the columns yields

$$\begin{array}{ccccc} & 0 & 0 & 1 & 1 \\ & 1 & 0 & 0 & 1 \\ + & 1 & 1 & 0 & 1 \\ = & 2 & 1 & 1 & 3 \end{array}$$

Sum 2113 is not the final result since each digit has to be taken modulo 2, or its remainder when dividing by 2. The Nim-sum is then $0111_b = 7$.

Bouton's Theorem. A nim position $n^k = (n_1, n_2, ..., n_k)$ is in $P$ if and only if the Nim-sum of its components is 0.

Proof. Let $Q$ be the set of positions with Nim-sum zero, and assume that $n^k = (n_1, n_2, ..., n_k)$ belongs to $Q$. Suppose there are sticks left and we remove them from pile $n_1$so that there remains $n_1' < n_1$ sticks. The Nim-sum of the resulting position is given by $n_1' \oplus n_2 \oplus ... \oplus n_k = n_1' \oplus n_1 \neq 0$ because each change of a binary digit going from $n_1$ to $n_1'$ causes the column sum to change modulo 2; of course it changes similarly taking the Nim-sum of $n_1'$ and $n_1$ (make sure you figure this out on paper). So any move from $Q$ leads to a position outside $Q$.

Proof (Continued) Assume next that $n^k = (n_1, n_2, ..., n_k) \notin Q$, and let $s = n_1 \oplus n_2 \oplus ... \oplus n_k \neq 0$. As $s$ is a binary expression its leftmost digit is unity, and consequently there must be an odd number of $n_i$, $i \in \{1, 2, ..., k\}$ such that its leftmost digit is unity. Choose one of these, call it $x_i$ and note that $x_i \oplus s < x_i$. This is because both $x_i$ and $s$ have unity at their leftmost position and the sum of that column is then 2 which is 0 modulo 2. Let a player remove $x_i - (x_i \oplus s)$ sticks from the $i^{th}$ pile causing $x_i$ to change to $x_i \oplus s$. The Nim-sum of the new position $(n_1, n_2, ...n_{i-1}, x_i \oplus s, n_{i+1}, ...n_k)$ is zero and thus it is in $Q$. Whenever a position lies outside $Q$ there is a move that results in a position that is in $Q$.

Proof (Continued) If the initial position is not in $Q$ the first player can always choose a move that leads to a position that is in $Q$. If the second player has any moves s/he necessarily moves from $Q$ to a position outside $Q$. Any position not in $Q$ is thus in $N$. Similar argument shows that if the initial position is in $Q$ the second player can guarantee a win. Any position in $Q$ is thus in $P$. QED

Knowledge about other players' intentions is central in game theory. This example hints at the interesting issues that arise when studying the epistemic aspects of the players' behaviour. The point is to differenciate between mutual knowledge and common knowledge.

Example2. There are twenty players in a room each with a red hat or a blue hat. The players can see each others' hats but they cannot see their own hats. There is also a clock that everyone can see. Every five minutes the players who can figure out the colour of their hats can tell it to an instructor, and the first player/s who figure out the colour correctly get a prize. A wrong guess involves a punishment so big that no-one wants to just take a chance and guess. The players are prevented from communicating with each other in any way.

- Assuming that eleven of the players have a red hat what happens?
- Nothing happens since every time five minutes has passed the players are exactly as wise as in the beginning of the game; no new information has appeared/been created.
- Since nothing happens the experiment is cancelled.
- Next day the experiment is repeated.
- But then the instructor announces to the players that at least one of them has a red hat.
- This is hardly big news to the players but in about an hour all the players with a red hat know that they have a red hat, and get the prize.

Let us see what happens. It is very useful to introduce some notation. It is also useful (as almost always) to study some simple cases first.

- We denote a knowledge operator by $K$.
- We denote the statement 'at least $i$ players are wearing a red hat' by $R_i$.
- Now we can formalise the statement 'player3 knows that at least 1 player is wearing a red hat' by $P3\,K\,R_1$.
- The simplest case (consistent with the instructor's announcement) is when exactly one player has a red hat; without loss of generality let the player be $P1$.

- Before the statement $P1$ sees only blue hats, and all other players see exactly one red hat.
- Formally, $P1 \neg K R_1$, and $Pi\, K R_1$ whenever $i \neq 1$.
- After the announcement everyone else's knowledge remains the same but now player1's knowledge is given by $P1\, K R_1$.
- And as player1 sees only blue hats s/he can figure out that s/he has a red hat.
- Assuming that the announcement is made when the minute hand is at noon and that the first time to tell one's knowledge to the instructor is five past, it takes five minutes once the game is over.

- Assume next that player1 and player2 have red hats.
- Before the announcement every player's state of knowledge is that there is at least one red hat; of course some players know that there are at least two red hats but this is of no significance since the players' inference starts from the instructor's announcement which is $R_1$.
- It is enough to focus on the inference of those players who have a red hat.
- Before the announcement $P1\,K\,R_1$ and $P2\,K\,R_1$, and also $P1\,\neg K\,(P2\,K\,R_1)$ and $P2\,\neg K\,(P1\,K\,R_1)$.
- After the announcement, however, $P1\,K\,(P2\,K\,R_1)$ and $P2\,K\,(P1\,K\,R_1)$.

- Consequently, each of the players can infer that if the other player sees only blue hats s/he will tell the colour of his/her hat to the instructor at five past.
- Because this is not the case at five past each player can infer that the other player saw exactly one red hat that must be his/hers.
- Then at ten past these two players tell the instructor that they have red hats.

- Assume that player1, player2 and player3 have red hats.
- Let us study the situation from the point of view of player1 because it is analogous to the other red hatters.
- Now player1 knows that each of the other red hatters knows that everyone knows that there is at least one red hat.
- But player1 does not know that player2 knows that player3 knows that there is at least one red hat.
- This is because from player1's point of view it is possible that s/he has a blue hat.
- In this case player2, who regards it as possible that s/he has a blue hat, cannot know that player3 knows that there is at least one red had (since player3 does not know the colour of his/her hat).
- Formally, $P1 \neg K\,(P2\,K\,(P3\,K\,R_1))$.

- After the announcement $R_1$ becomes common knowledge, or $P1\,K\,(P2\,K\,(P3\,K\,R_1))$.
- Now each of the red hatters regards it as possible that there are only two red hats.
- In this case the two red hatters would tell the instructor the colours of their hats at ten past.
- But this does not happen, and so the three red hatters know that there must be three red hats (the only other possibility), and they tell the colours to the instructor at fifteen past.
- Continuing the logic, at 55 past the eleven red hatters of the original problem know the colour of their hats and the game ends.

Example3. Gale-Shapley algorithm

- Consider marriage market where the objective is to match men and women in pairs.
- Design a game that has desirable features.
- Let there be $n$ men ja $n$ women.
- Each player has a preference ordering (ranking) over the potential partners.
- This is what they tell the market maker.
- Denote men by lower case and women by upper case letters.
- An example is given below.

Men's preferences

$$\begin{array}{rcccc} & A & B & C & D \\ a \rightarrow & 3 & 4 & 1 & 2 \\ b \rightarrow & 2 & 3 & 4 & 1 \\ c \rightarrow & 1 & 2 & 3 & 4 \\ d \rightarrow & 3 & 4 & 2 & 1 \end{array}$$

Women's preferences

$$\begin{array}{ccccc} & A\downarrow & B\downarrow & C\downarrow & D\downarrow \\ a & 1 & 1 & 3 & 2 \\ b & 2 & 2 & 1 & 3 \\ c & 3 & 3 & 2 & 1 \\ d & 4 & 4 & 4 & 4 \end{array}$$

Study one possible solution.

$$\begin{cases} a & A & & 3,1 \\ b & B & & 3,2 \\ c & C & & 3,2 \\ d & D & & 1,4 \end{cases}$$

where the first number is the man's ranking of the woman and the second the woman's ranking of the man.

- This is not a stable matching because $D$ can propose $b$ that they form a pair.
- $D$ ranks $b$ the third and $b$ ranks $D$ the first.

**Definition.** A stable matching is sucht that there is no man and woman who consider each other better than their current partners.

- The following pairs form a stable matching

$$\begin{cases} a & C & 1,3 \\ b & D & 1,3 \\ c & A & 1,3 \\ d & B & 4,4 \end{cases}$$

- There are two problems: i) Figure out whether a stable matching exists, and ii) how to find a stable matching.
- Gale-Shapley-algorithm takes care of both problems.

1. Each man makes a proposal to a woman who is number 1 in his ranking. Women who get at least two proposals select the highest ranking man on a waiting list and tell other men never to contact her again.
2. Each rejected man makes a proposal to a woman who is number 2 in his ranking. The women choose the highest ranking man on the waiting list and tell others not to make a proposal any more.The men on the waiting list from the previous round are regarded as proposers.
3. Each rejected man makes a proposal to a woman whom he ranks next highest. The women choose the highest ranking man on the waiting list and tell others not to make a proposal any more.
4. Continue as above.

- Does this process always stop?
- Does it produce a stable matching?
- The answer is positive to both questions.
- Let us first study an example.

Men's preferences

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $a \rightarrow$ | 1 | 2 | 3 | 4 |
| $b \rightarrow$ | 1 | 4 | 3 | 2 |
| $c \rightarrow$ | 2 | 1 | 3 | 4 |
| $d \rightarrow$ | 4 | 2 | 3 | 1 |

Women's preferences

| | $A\downarrow$ | $B\downarrow$ | $C\downarrow$ | $D\downarrow$ |
|---|---|---|---|---|
| $a$ | 3 | 3 | 2 | 3 |
| $b$ | 4 | 1 | 3 | 2 |
| $c$ | 2 | 4 | 4 | 1 |
| $d$ | 1 | 2 | 1 | 4 |

- First round produces

$$\left\{ \begin{array}{cc} a, b* & A \\ c & B \\ & C \\ d & D \end{array} \right.$$

where the asterisk denotes a rejectedd man. No-one proposes woman $C$.

- Second round produces

$$\left\{ \begin{array}{cc} a & A \\ c & B \\ & C \\ d*, b & D \end{array} \right.$$

- Third round produces

$$\begin{cases} a & A \\ c*, d & B \\ & C \\ b & D \end{cases}$$

- Fourth round produces

$$\begin{cases} a*, c & A \\ d & B \\ & C \\ b & D \end{cases}$$

- Fifth round produces

$$\begin{cases} c & A \\ d, a* & B \\ & C \\ b & D \end{cases}$$

- Sixth round produces

$$\begin{cases} c & A \\ d & B \\ a & C \\ b & D \end{cases}$$

**Theorem.** Gale-Shapley-algorithm ends after a finite number of rounds.

**Proof.** If a woman has several proposals there exists a woman without any proposals because there are equal numbers of men and women. If a woman ever gets a proposal she has a man on a waiting list from there on. But as long as there is a woman without a proposal there is a man who keeps on making proposals, and the algorithm necessarily ends in a state where each woman has a proposal. QED

**Theorem.** Gale-Shapley-algorithm terminates in a stable matching.

**Proof.** Assume that in the final stage there exist pairs $(r, R)$ and $(s, S)$ such that $r$ regards $S$ as better than his current partner $R$. This means that at some stage $r$ has proposed to $S$. But then $S$ has rejected $r$ at some point. All men on the waiting list of $S$ are better than $r$ after she has rejected $r$.

- Algorithm can work also so that women propose to men.
- When men propose one gets a man-optimal matching.
- When women propose one gets a woman-optimal matching.
- If these coincide there exists only one stable matching.
- When men propose their optimal strategy is to propose according to their preferences, but in some cases women can benefit from strategic behaviour.
- More insight to the G-S-algorithm can be found in Gura and Maschler 2008 "Insights into game theory".

## Exercise

Consider the following preferences.

| | $A$ | $B$ | $C$ | $D$ |
|---|---|---|---|---|
| $a \rightarrow$ | 4 | 2 | 1 | 3 |
| $b \rightarrow$ | 2 | 1 | 3 | 4 |
| $c \rightarrow$ | 3 | 1 | 4 | 2 |
| $d \rightarrow$ | 2 | 4 | 1 | 3 |

| | $A \downarrow$ | $B \downarrow$ | $C \downarrow$ | $D \downarrow$ |
|---|---|---|---|---|
| $a$ | 1 | 2 | 4 | 2 |
| $b$ | 2 | 4 | 2 | 1 |
| $c$ | 3 | 1 | 1 | 3 |
| $d$ | 4 | 3 | 3 | 4 |

Determine the stable matchings when men propose and when women propose.