Game theory lecture 5

October 5, 2013

Games in extensive form

- In normal form games one can think that the players choose their strategies simultaneously.
- In extensive form games the sequential structure of the game plays a central role.
- In this section we assume that the players' information is perfect, i.e., that each player knows all the actions taken when it is his/her turn to take an action (or to move).
- The standard way to depict small extensive form games is to use game trees. The definition of an extensive form game is, though, quite complicated.

Definition. An extensive form game consists of the set of players, the set of terminal histories, a player function, and preferences for the players. Formally, an extensive form game $\langle N, H, P, (u_i) \rangle$consists of the set of players N, the set of sequences H called the set of histories, the player function P that assigns to each history a member of N, and a utility function u_i for each member in N. The set of histories satisfies i) $\emptyset \in H$, ii) if $\left(a^k\right)_{k=1}^{K} \in H$ then $\left(a^k\right)_{k=1}^{L} \in H$ for $L < K$, iii) if $\left(a^k\right)_{k=1}^{\infty}$ satisfies $\left(a^k\right)_{k=1}^{K} \in H$ for all K then $\left(a^k\right)_{k=1}^{\infty} \in H$.

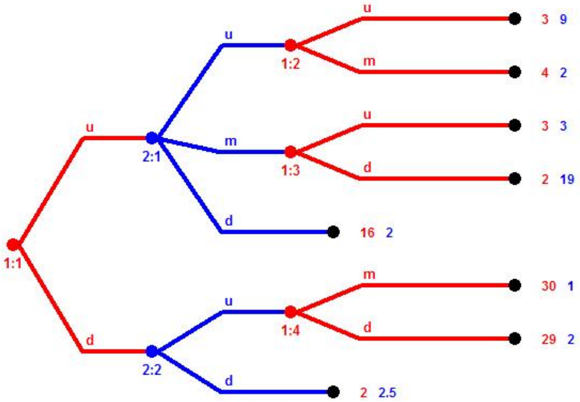
Games in extensive form

- The interpretation is that each history consists of actions of players.
- Terminal histories are such that there is no a^{K+1} such that $\left(a^k\right)_{k=1}^{K+1} \in H$ after some K, or the history is infinitely long.
- The player function assigns the empty set to finite terminal histories.
- If a game has only finitely long histories, and the number of terminal histories is finite, the game is called finite.
- Otherwise it is infinite.

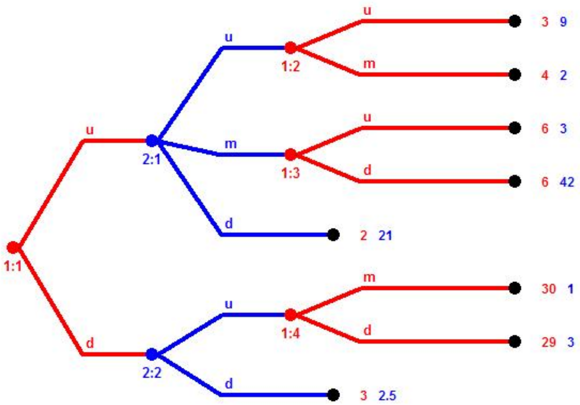
Games in extensive form

- There is a signicant difference between games that warrant infinitely long histories and games that do not.
- A powerful method of backward induction can be used to find at least some Nash-equilibria in the latter class of games.
- Backward induction is an algorith in which one starts from the end of the game and works backwards towards the beginning of the game.
- One starts with subgames (to be defined) of length (to be defined) one, and determines the optimal actions of the players.
- Then one considers subgames of length two (knowing what happens in the following subgames of length one), and repeats the procedure.
- This way one tracks down all the optimal actions of the players at all their decision nodes.
- The following three examples illustrate the procedure.

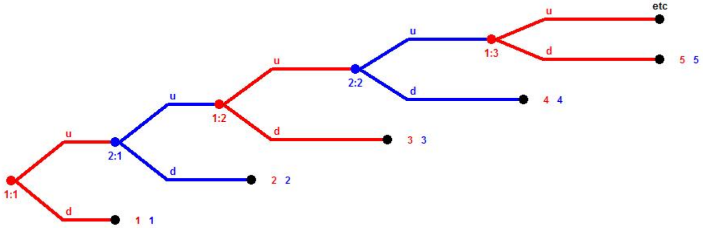
Example1. A game where backward induction bites



Example2. A game where backward induction does not bite

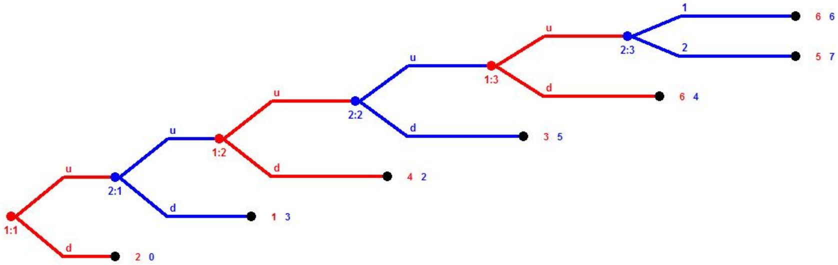


Example3. Another game where backward induction fails



Games in extensive form

Example4. A game where backward induction works too well



Games in extensive form

- In the second example backward induction does not yield a unique answer because at some decision nodes a player is indifferent between actions.
- In the third example the problem is not pay-offs but in an infinitely long game there is no last point from where to start the backward induction.
- As it turns out backward induction is not strong enough to provide a unique solution to extensive form games in general.
- For this reason we have to resort to Nash-equilibrium as a solution concept.
- But to do that we first need to define strategies in extensive form games.
- A strategy is a tricky concept, and it is good to bear in mind that a strategy is a complete plan of how to play in each decision node of the game.
- Even in decision nodes that are not reachable according to the strategy.

Definition. Strategy A strategy of player i in an extensive form game $\langle N, H, P, (u_i) \rangle$ is a function s_i that assigns to each such history h that $P(h) = i$ and element from $A_i(h)$, the set of actions available to player i after history h.

- Notice that the above denition is a little loose.
- To define strategies completely unambiguously one needs more machinery than we are willing to cover in this course (see e.g. Foundations of Non-Cooperative Game Theory by Ritzberger for details).

Games in extensive form

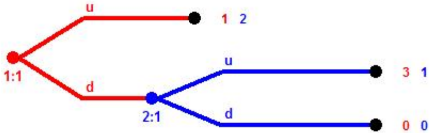
- Defining a strategy of player 1 in the first game above requires that we agree on the order in which his/her decision nodes (note that there is one-to-one correspondence between decision nodes and histories) are considered.
- Let us agree that we address them from left to right, and downwards.
- Thus, one strategy would be (d, u, u, m).
- Player 1's set of pure strategies consists of 24 elements, while that of player 2 consists of 23 elements.
- The strategies got by backward induction are $(u, m, u, m; m, d)$ where player 1's strategy is given first.

- Strategies induce a terminal history, and each terminal history is associated with a pay-off for the players.
- The Nash-equilibrium of an extensive form game can be dened in the standard way.

Definition. Nash equilibrium A Nash-equilibrium $s \in \times_{j=1}^{n} S_j$ of an extensive form game $\langle N, H, P, (u_i) \rangle$ is such that for each player i $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ for any other strategy $s_i' \in S_i$.

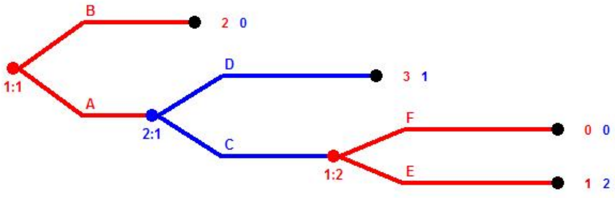
- It is convenient to postulate that terminal histories are associated with outcomes, and that the utilities associated with strategies are actually the utilities from the outcomes that the strategies induce.

Example5. A game with several equilibria



- One equilibrium is given by $(u; d)$ and another by $(d; u)$.
- The interpretation is that the first equilibrium is based on a non-credible threat, and for this reason it does not look good to us.
- But it still satises the requirements of the Nash- equilibrium.

Example6. Game on page 160 of Osborne



- Player 1 has two decision nodes so that his/her strategy must specify what to do in each. Player 1's set of strategies is thus $S_1 = \{(AE), (AF), (BE), (BF)\}$ and player 2's set of strategies is $S_2 = \{C, D\}$.
- The pure Nash-equilibria of the game are $(((A, F); D), ((B, F); C), ((B, E); C))$.

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- One can find these equilibria by constructing a normal form game that corresponds to the extensive form game as below

	C	D
AE	$1,2$	$3,1$
AF	$0,0$	$3,1$
BE	$2,0$	$2,0$
BF	$2,0$	$2,0$

- There is some reduncancy and the so called reduced normal form is given by

	C	D
AE	$1,2$	$3,1$
AF	$0,0$	$3,1$
B	$2,0$	$2,0$

- The examples above demonstrate that the concept of Nash-equilibrium looks sometimes unsatisfactory as it does not take into account the sequential structure of the game.
- For this reason we introduce a refinement of Nash-equilibrium.
- In words a Nash-equilibrium is subgame perfect if in each decision node the strategy assigns an action that is optimal against other players' actions.

Definition. Subgame perfect Nash-equilibrium
Let $\Gamma = \langle N, H, P, (u_i) \rangle$ be an extensive form game. For any non-terminal history h the subgame $\Gamma(h)$ following the history is the extensive form game $\Gamma(h) = \langle N, H(h), P(h), (u_i(h)) \rangle$ where $h' \in H(h)$ if and only if $(h, h') \in H$, $P(h)(h') = P((h, h'))$ and $u_i(h)(h') = u_i((h, h')))$.

- In example 5 there are three subgames as the game itself is a subgame following the history $\oslash$.

- Consider strategy s_i of player i and history h.
- Denote the strategy induced by s_i after history h by $s_i|_h$.

Definition. Subgame perfect Nash equilibrium
A strategy profile s^* in an extensive form game $\Gamma = \langle N, H, P, (u_i) \rangle$ is a subgame perfect Nash- equilibrium if, for each player i, and each non-terminal history $h \in H$ for which $P(h) = i$ the following holds $u_i\left(s_i^*|_h, s_{-i}^*|_h\right) \geq u_i\left(s_i, s_{-i}^*|_h\right)$ for all strategies s_i in the subgame $\Gamma(h)$ and for all players $i \in N$.

- Another way of saying this is that a subgame perfect Nash-equilibrium of an extensive form game has to constitute a Nash-equilibrium in each subgame of the extensive form game.
- In example 5 only $(d; u)$ is a subgame perfect Nash-equilibrium.

Games in extensive form

- Finding subgame perfect equilibria in finitely long games can be done by backward induction.
- The induction happens on the length of the subgames where the length of a game is its longest history.
- One first finds the subgames of length one and determines their Nash- equilibria.
- Then one fixes one Nash-equilibrium for each of these subgames and considers subgames of length two.

Games in extensive form

- The players whose turn it is to make a choice in these games know what happens after their choice (in subgames of length 1).
- Thus, one can determine the Nash-equilibrium choices of subgames of length two.
- One keeps on going like this until one reaches the initial node.
- Whenever there are several optimal choices, or Nash-equilibria, in a subgame one has to fix each of the Nash-equilibria in turn to find all subgame perfect Nash-equilibria.
- In example 2 the subgame perfect Nash-equilibria (SPE from now on) in pure strategies are $(d, m, u, m; d, d)$ and $(u, m, d, m; m, d)$.

Proposition. The procedure of backward induction finds all pure strategy SPE in finitely long extensive form games.

- It is clear that backward induction never gives an empty set as a result and consequently we can state

Proposition. In finitely long extensive form games an SPE always exists.

Games in extensive form

- Note that so far we have not said anything about mixed strategies.
- The idea is the same as in normal form games but mixed strategies are not so relevant in extensive form games as in normal form games.
- In example 2 player 1 could easily mix between actions u and d in his/her third decision node as s/he is indifferent between them.
- That would introduce SPE where in some subgames there is mixing.

- The objects of choice in extensive form games are the complete game plans, strategies, and one can easily consider mixing between them.
- There is a remarkable theorem by Kuhn (1953) that provides the following equivalence: To each mixed strategy corresponds (in expected utility as well as in expected outcome terms) so called behavioural strategy where the players mix (in an appropriate way) in their decision nodes, and vice versa.