Lecture 8

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Example of mixed strategy equilibrium

- There are three buyers with unit demands.
- Two are of low valuation $v < 1$.
- One is of high valuation 1.
- There are two identical sellers with one unit each.
- Sellers are in the same location and choose price before buyers arrive.
- Buyers arrive in random order.

Example of mixed strategy equilibrium

- Both sellers are expected to ask price v.
- Assume a seller deviates and asks price 1; this is the only deviation that has to be checked!
- S/he makes a sale with probability $\frac{2}{3}$, i.e., if the high-valuation buyer does not arrive first.
- As long as $\frac{2}{3} \leq v$ the deviation is not profitable.
- Then both sellers asking price v is a pure strategy Nash-equilibrium.

Example of mixed strategy equilibrium

- Assume that $\frac{2}{3} \leq v$, or deviation from (v, v) to unity is profitable.
- Then the symmetric equilibrium is in mixed strategies.
- Can the equilibrium strategy be continuous?
- Then the lowest value in the support of the mixed strategy must be v; why?
- Asking v would generate pay-off v.

Example of mixed strategy equilibrium

- The highest value in the support of the mixed strategy must be unity.
- Unity must then generate pay-off v, too.
- A seller who asks unity gets pay-off zero, as v is chosen with probability zero, and low-valuation buyers cannot buy either good.
- This is impossible.

Example of mixed strategy equilibrium

- There must be a mass point at price v.
- With probability γ a seller asks price v.
- With probability $1-\gamma$ s/he uses a continuous mixed strategy F with support $[c,1]$ where $c > v$.
- Notice that c must be strictly bigger than v since there is a discrete jump in the selling probability at prices $p > v$.

Example of mixed strategy equilibrium

- Asking price v results in trade with certainty.
- Asking price 1 results in trade only if the other seller asks price v which takes place with probability γ and the first arriving buyer is a low-valuation one.
- This happens with probability $\frac{2}{3}\gamma$.
- Asking price unity yields consequently pay-off $\frac{2}{3}\gamma$.
- In equilibrium $v = \frac{2}{3}\gamma$ from which we get $\gamma = \frac{3}{2}v$.

Example of mixed strategy equilibrium

- Asking price $x \in [c, 1)$ results in trade if the other seller asks price v and the first arriving buyer is a low-valuation one, or if the other seller asks a price higher than x.
- This happens with probability $\frac{2}{3}\gamma + (1-\gamma)(1-F(x))$.
- Consequently, we must have

$$x\left[\frac{2}{3}\gamma + (1-\gamma)(1-F(x))\right] = v$$

- From this we can solve

$$F(x) = \frac{(2-v)x - 2v}{(2-3v)x}$$

Example of mixed strategy equilibrium

- Inserting $x = c$ and $F(c) = 0$ we can solve $c = \frac{2v}{2-v}$.
- What we learn is that in equilibrium the low-valuation buyers do not get any surplus because there are more buyers than sellers.
- Also, there is a mass point in the mixed strategy at the low-valuation buyers' valuation, and there is a gap between v and the lower bound of the support of the mixed strategy c.
- Most importantly, the equilibrium features inefficiency due to the random arrival of the buyers.

An application to incomplete contracts

- We take it as given that not all contingencies can be contracted upon.
- The interesting case is when some contingencies/states are not verifiable but the parties to a contract still observe the states.
- Then contracts upon these cannot be taken into the court.
- Assume two persons A and B who want to engage in a common enterprise; we call it a project.
- The project consists of A and B and physical capital.
- The surplus the project generates depends on how much A and B invest in it.

An application to incomplete contracts

- We assume that the investments take place sequentially, the parties observe each others' investment but they are not verifiable.
- Denote A's and B's investments by a and b.
- Assume that these also signify the costs of investments.
- A and B must agree on the ownership relations and how the surplus is to be divided between them.
- Before going to this let us study the situation, and specify the details of surplus generation.

An application to incomplete contracts

- The game consists of three stages.
- In the first A invests.
- In the second B invests.
- In the third the surplus is realised as $\sqrt{a} + \sqrt{b} + \sqrt{ab}$.
- The problem from the players' point of view is that even if they make big investments they cannot be remunerated for this as only the surplus is verifiable.

An application to incomplete contracts

- The socially optimal solution is got as a solution to

$$max_{a,b}\sqrt{a}+\sqrt{b}+\sqrt{ab}-a-b$$

- The first order conditions are

$$\frac{1}{2}\frac{1}{\sqrt{a}}+\frac{1}{2}\frac{1}{\sqrt{ab}}b-1=0$$

$$\frac{1}{2}\frac{1}{\sqrt{b}}+\frac{1}{2}\frac{1}{\sqrt{ab}}a-1=0$$

- Symmetric solution is given by $a^* = b^* = 1$.

An application to incomplete contracts

- We also see that once A has invested the socially optimal investment of B is given by

$$b^*(a) = \frac{1}{4}\left(1+\sqrt{a}\right)^2$$

- But this happens only if B gets the whole surplus.
- Let us see what happens if A owns the project or in this case the physical capital.
- Ownership means that A can contract who can use the capital; A has all the rights that s/he has not contracted away.

An application to incomplete contracts

- Whichever way the ownership is allocated, e.g., to A, or to B, or jointly to both the parties do not invest optimally.
- This is because at the time the parties divide the surplus $\sqrt{a}+\sqrt{b}+\sqrt{ab}$ their investments are sunk.
- For instance, if they divide the surplus in half each gets only half of his/her contribution while having paid all of it.
- In this case B would solve the following program

$$max_b \frac{1}{2}\left(\sqrt{a}+\sqrt{b}+\sqrt{ab}\right) - b$$

- First order condition is

$$\frac{1}{2}\left(\frac{1}{2}\frac{1}{\sqrt{b}}+\frac{1}{2}\frac{1}{\sqrt{ab}}a\right)-1=0$$

from which one can solve

$$b=\frac{\left(1+\sqrt{a}\right)^2}{16}<b^*(a)$$

An application to incomplete contracts

- The following contract induces efficiency.

Contract. In the beginning A owns the capital but B has an option to buy the capital at price p^* after A has invested but before the surplus is realised.

- The crucial issue is to determine the correct level of the price p^*.

Claim. At price $p^* = 2$ A and B make socially optimal investments.

Proof. If B uses the option s/he gets $\sqrt{a} + \sqrt{b} + \sqrt{ab} - b - p^*$. This expression is maximised at $b^*(a) = \frac{1}{4}\left(1 + \sqrt{a}\right)^2$. If B does not use the option his/her utility is zero. So B uses the option only if

$$\sqrt{a} + \sqrt{\frac{1}{4}\left(1 + \sqrt{a}\right)^2} + \sqrt{a\frac{1}{4}\left(1 + \sqrt{a}\right)^2} - \frac{1}{4}\left(1 + \sqrt{a}\right)^2$$

$$\geq p^* = 2$$

An application to incomplete contracts

The LHS of the inequality is increasing in a, so B uses the option only if A invest sufficiently. Or course, A invest just what is needed and nothing more. At value $a = 1$ there is equality and B invests $b = 1$. This generates utility 1 to A. If A invests less his/her utility is $\sqrt{a} - a < 1$.QED