OPTIMAL SEARCH and ONE-SIDED SEARCH

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We first cover the papers by Diamond (1971) and Weitzman (1979).

Diamond's model

- In the Diamond's paradox there are buyers who all have a unit demand, and value a homogeneous good at unity.
- There are also sellers who sell the good.
- The prices are available only when a buyer enters a store.
- Visiting a store costs $c > 0$.
- The paradox is that in equilibrium the price of the good is unity regardless of the number of sellers.

- The logic goes as follows.
- Suppose that in equilibrium some sellers asked different prices, and let the lowest of them be $p < 1$.
- A buyer who contacts this seller cannot expect to find a lower price.
- Consequently, the seller could equally well ask price $p + c$.
- The buyer would not find it profitable to go to another store.
- Assume that all sellers ask the same price $q < 1$.
- Now any seller could ask price, say, $q + min\left\{\frac{1}{2}(1-q), c\right\}$, and increase his/her profit.

- This reasoning a little loose since the strategic situation is not carefully modelled.
- It is also based on backward induction type argument.
- Consider a situation with just one buyer and two sellers.
- The buyer can approach either seller, and visiting a seller costs $0,1$.
- The buyer's strategy is to refuse any trade where the seller asks more than $2/3$, and accept all others.
- Assume that each seller asks price $2/3$.

- This is a Nash-equilibrium when the buyer contacts each seller with equal probability.
- It is not, however, a sub-game perfect equilibrium.
- Notice that in the original set-up of Diamond (1971) the buyers search optimally.
- There would be no contacts at all in equilibrium unless one assumes that the first contact is free.
- To generate something more interesting one has to assume more heterogeneity or a more interesting informational setting.

Weitzman's model

- There is an exogenously given set of alternatives named by positive integers $I = \{1, 2, ..., n\}$.
- The worth of alternative i is stochastic and given by distribution F_i.
- The cost of investigating alternative i is c_i, and it takes time t_i for the worth to be revealed.
- The agent discounts future at rate r.

Example

- Interest rate is $r = 0,1 = 10\%$, and there are two alternatives α and ω summarised in the table

project	α		ω	
cost	15		20	
duration	1		2	
reward	100	55	240	0
probability	0,5	0,5	0,2	0,8

- Investigating alternative α yields expected profit

$$-15 + \frac{1}{1,1}[0,5 \cdot 100 + 0,5 \cdot 55] = 55,5$$

while investigating alternative ω yields expected profit

$$-20 + \left(\frac{1}{1,1}\right)^2 [0,2 \cdot 240 + 0,8 \cdot 0] = 19,7$$

- Suppose α is investigated first. If it turns out that its value is 55, then the agent investigates ω, too.
- The expected value from this is

$$-20+\left(\frac{1}{1,1}\right)^2[0,2\cdot 240+0,8\cdot 55]=56$$

- If it turns out that its value is 100 investigating ω yields

$$-20+\left(\frac{1}{1,1}\right)^2[0,2\cdot 240+0,8\cdot 100]=85,8$$

and investigation does not make sense.

- Consequently, starting with α optimally yields

$$-15+\frac{1}{1,1}\left[0,5\cdot 100+0,5\cdot\left(-20+\left(\frac{1}{1,1}\right)^2[0,2\cdot 240+0,8\cdot 55]\right)\right]$$
$$=55,9$$

- Suppose ω is investigated first.
- If its value turns out zero it is clear that it is optimal to develop α.
- If its value turns out 240 it is clear that it is optimal not to investigate α.

- Starting with ω, and behaving optimally yields

$$-20+\left(\frac{1}{1,1}\right)^2\left[0,2\cdot 240+0,8\cdot\left(-15+\frac{1}{1,1}[0,5\cdot 100+0,5\cdot 55]\right)\right]$$

$$=56,3$$

- The point is that one has to take into account the optimal revelation of information, too.

- Let us return to the general case.
- Whenever the agent decides to stop searching s/he can choose the best alternative so far; this is search with recall.
- Costs are being paid while search takes place; the reward is got only after the search is over.
- Assume that the agent has investigated alternatives in set S, and at the present the highest of them is y.
- The sufficient statistic to the agent's optimal behaviour is (S^c, y); it is immaterial to know in which order the agent has investigated the alternatives in S.

- By the principle of optimality for dynamic programming the agent's value function at state (S^c, y) is given by

$$\Psi(S^c, y) =$$

$$max\left\{y, max_{i \in S^c}\left\{-c_i + \beta_i\left[\Psi(S^c - \{i\}, y)\int_{-\infty}^{y} dF_i(x_i) + \int_{y}^{\infty} \Psi(S^c -\right.\right.\right.$$

where $\Psi(\text{Ø},\text{x}) = x$ and $\beta_i = e^{-rt_i}$.

- Evaluating the value function is hard!

- The aim is to find an index that is a sufficient statistic for the worth of each alternative.
- Then one need not worry about durations, costs, distributions.
- Assume that at some point of the search the best investigated alternative is worth z.
- If the agent investigates alternative i s/he expects to get

$$-c_i + \beta_i \left[z \int_{-\infty}^{z} dF_i(x_i) + \int_{z}^{\infty} x_i dF_i(x_i) \right]$$

- The agent is indifferent between investigating alternative i and taking z if and only if

$$c_i = \beta_i \int_z^{\infty} (x_i - z)\, dF_i(x_i) - (1 - \beta_i)\, z \tag{1}$$

- The very z that satisfies expression (1) is called the reservation value/price of alternative i.

Theorem. At any stage of the search, if the agent has to investigate an alternative, the optimal policy/strategy is to investigate the alternative with the highest reservation value/price.

Theorem. At any stage of the search it is optimal to cease the search when the maximum worths of the investigated alternatives exceeds the maximum of the remaining alternatives' reservation values/prices.

One-sided search

- Let us consider some examples of search, given in the labour market setting, to familiarise ourselves with the techniques.
- A solitary worker is looking for a job sequentially.
- S/he knows that the distribution of wages is given by F on $[\omega, \Omega]$; let f be the corresponding density function.
- S/he can get one job offer per period and discounts future with factor $\delta = \frac{1}{1+r} \in (0, 1)$.
- If the agent has passed an offer s/he cannot return to it.

- Assume that s/he has got a wage offer w.
- Now his/her value function is given by

$$V(w) = max\left\{\delta V, \frac{w}{1-\delta}\right\}$$

where the first term is the worker's expected pay-off from searching (optimally), and the second is the life-time payment from accepting offer w.

- The first term is given by

$$V = \int_{\omega}^{\Omega} V(x)f(x)dx$$

- Optimal behaviour is captured by a treshold-rule T

$$V = \int_{\omega}^{T} \delta V f(x) dx + \int_{T}^{\Omega} \frac{x}{1-\delta} f(x) dx$$

$$= \delta V F(T) + \int_{T}^{\Omega} \frac{x}{1-\delta} f(x) dx$$

- From this we get the expected life-time utility, or the value function,

$$V = \frac{1}{1-\delta F(T)} \int_{T}^{\Omega} \frac{x}{1-\delta} f(x) dx$$

- The reservation level is determined by indifference

$$\frac{T}{1-\delta} = \delta V = \frac{\delta}{1-\delta F(T)} \int_T^{\Omega} \frac{x}{1-\delta} f(x)dx$$

which is equivalent to

$$T = \frac{\delta}{1-\delta F(T)} \int_T^{\Omega} xf(x)dx$$

- From this equation we get a more informative expression by multiplying by $1-\delta F(T)$ and subtracting δT from both sides

$$T = \frac{\delta}{1-\delta} \int_T^{\Omega} (x-T)f(x)dx$$

- The problem here is that the wage distribution is exogenous, while it should be the result of the employers' optimising behaviour.
- One must come up with a credible, and tracktable, story about wage determination.

- Let us develop this model a little by assuming that the wages are given by F, and that unemployed workers get an unemployment benefit b, and that there is an exogenous separation rate s.
- Now we need to determine the value functions of the unemployed V_u as well as those who are employed at wage w, $V_e(w)$.
- The Bellman equations are given by

$$V_u = \delta \left\{ b + \int_{\omega}^{\Omega} max \left\{ V_e(x), V_u \right\} f(x) dx \right\}$$

and

$$V_e(w) = \delta \left\{ w + sV_u + (1-s) V_e(w) \right\}$$

- In formulating the value functions one has to be careful about the order of events within a period.

- Here the value functions are evaluated in the end of a period.
- Because of that everything on the right-hand-side is discounted.
- In the beginning of a period the agent gets either the unemployment benefit or the wage.
- Then s/he gets a wage offer if s/he is unemployed and decides whether to accept or to continue search.
- If s/he is employed she loses the job with probability s, and becomes unemployed.
- With the complementary probability s/he continues in his/her job.

- Again it is clear that the unemployed use a treshold strategy where they only accept wage offers above R.
- Again R is given by the condition that wage offer of this size makes the unemployed indifferent between accepting and continuing search.
- Many times the value functions above are given in the so called asset valuation form.
- Remembering that $\delta = \frac{1}{1+r}$, multiply both sides by $1+r$ and subtract V_u (or $V_e(w)$) from both sides.
- This produces

$$rV_u = b + \int_{\omega}^{\Omega} max\left\{V_e(x) - V_u, 0\right\} f(x)dx \qquad (2)$$

and

$$rV_e(w) = w + s\left[V_u - V_e(w)\right] \qquad (3)$$

- There is no standard way of solving the Bellman equations (most times one cannot do it).
- Now it is natural to start from (3) as it is linear

$$V_e(w) = \frac{w + sV_u}{r + s} \tag{4}$$

- The point is to notice that this is strictly increasing in w; this shows the optimal strategy consists of threshold R.
- The treshold is determined by indifference

$$V_e(R) = V_u$$

and substituting this to (4) we get

$$V_u = \frac{R}{r}$$

- Now we use this piece of information in equation (2) to get

$$R = b + \int_{\omega}^{\Omega} max\left\{\frac{x-R}{r+s}, 0\right\} f(x)dx \tag{5}$$

- As long as $x < R$ the integral with respect to the first argument of the max-function gives something less than zero.
- Thus, (5) can be simplified to

$$R = b + \int_{R}^{\Omega} \frac{x-R}{r+s} f(x)dx \tag{6}$$

- Simplifying the above some more notice that $\int_R^\Omega \frac{R}{r+s} f(x)dx = \frac{R}{r+s}(1-F(R))$ and by partial integration $\int_R^\Omega xf(x)dx = |_R^\Omega xF(x) - \int_R^\Omega F(x)dx$ $= \Omega - RF(R) - \int_R^\Omega F(x)dx$.

- Inserting these data into (6) one gets

$$R = b + \frac{1}{r+s}\left[\Omega - RF(R) - \int_R^{\Omega} F(x)dx - R(1-F(R))\right]$$

which is equivalent to

$$R = b + \frac{1}{r+s}\int_R^{\Omega}(1-F(x))dx \qquad (7)$$

since $\Omega - R = \int_R^{\Omega} 1dx$.

- In more complicated cases the basic idea is the same as above.
- One should notice that even in this simple case one does not get an explicit expression for the reservation wage but it is implicitly defined by (7).
- In this case the reservation wage is unique; the LHS is zero at $R = 0$, while the RHS is strictly positive, and both sides are continuous in R.
- The LHS is increasing in R and the RHS decreasing, and at $R = \Omega$ the LHS is strictly positive, while the RHS is b.
- As long b is not too large the reservation wage is strictly positive.
- Exercise. Totally differentiate (7) to get comparative statics results.

- To make the above setting a (primitive) labour market let us assume that there is a continuum α of workers.
- Denote the fraction of the unemployed by u.
- The idea is the equate the flows into unemployment and away from it.
- Each period the measure of employed who lose their jobs is given by $s(1-u)\alpha$, and the measure of unemployed who find a job is given by $u\alpha(1-F(R))$.
- In a steady state these flows must be equal which give us

$$u = \frac{s}{s+1-F(R)}$$

- This approach is very partial equilibrium type.
- There is no reasonable way to address the question about the number of vacancies.
- To make progress one has to consider a two-sided search model where employers and workers are modelled in the same detail.