Diamond's coconut model

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- There are no wages but there is heterogeneity in rewards.
- The model is in continuous time.
- There is an island with a continuum of agents, say, the unit interval.
- The only thing to consume is coconuts which grow in palms that vary in height.
- The higher a coconut is the more costly it is to get.
- All the coconuts are the same, though, and if an agent consumes at time t his/her utility is $u(t) = y$ and otherwise zero.

- If an agent collects a coconut at time t the cost is $c(t)$.
- His/her utility at time τ is given by

$$V(\tau) = \int_{\tau}^{\infty} e^{-rt} (u(t) - c(t))\, dt$$

- The cost of getting a coconut is given by a distribution function F with support $[\underline{c}, \bar{c}]$.
- To facilitate trading we make an extreme assumption that an agent cannot consume the coconut that s/he has him/herself collected.
- To consume the agents have to find a partner who also has a coconut and this does not happen instantaneously.
- Let us further simplify by assuming that the agents can carry around at most one coconut.

- Meetings are not modelled explicitly but it is assumed that when the measure of agents looking for a partner is e then they meet at a flow rate $b(e)$, i.e., the expected time until the first meeting is exponentially given with parameter $b(e)$.
- Another way to say this is that meetings happen with Poisson-rate $b(e)$. Heuristically this means that within a short interval $\triangle t$ the probability of meeting exactly one agent is given by $b(e)\triangle t$, and the probability of meeting more than one agent is of type $o(\triangle t)$, i.e., negligible.

We make an assumption that amounts to positive externalities in the (unmodelled) meeting technology

Assumption. $b'(e) > 0$.

- Coconut trees are encountered with an exogenous Poisson-rate a.

- The optimal behaviour of an agent depends on the cost of getting a coconut and the measure of agents looking for partners.
- Heuristically, if the cost is low and there are many agents around an agent finds it profitable to get a coconut.
- We model the strategy $p : [\underline{c}, \bar{c}] \times [0,1] \to [0,1]$ as a probability of acquiring the coconut.[1]
- With these assumptions we can formulate the value-functions, or Bellman equations, of the agents.

[1] We could also let the strategy to depend on time but for simplicity we focus on a steady state.

- They can be in two states, namely they may possess a coconut and wait for a partner or they can wait to find an opportunity to get a coconut.
- Denote the expected utility in the former state by V_E and in the latter state by V_U.
- Let us derive the expression for V_E explicitly, and leave the other one as an exercise

$$
\begin{aligned}
V_E(\tau) &= \int_\tau^\infty e^{-rt}[u(t) - c(t)]\,dt \\
&= [\Delta\tau b(e(\tau)) + o(\Delta\tau)][y + V_U(\tau)] \\
&+ [1 - \Delta\tau b(e(\tau)) - o(\Delta\tau)]\, e^{-r\Delta\tau} \int_{\tau+\Delta\tau}^\infty e^{-rt}[u(t) - c(t)]\,dt
\end{aligned}
$$

- The first equality is just the definition where one has to understand that $u(t)$ and $c(t)$ get a strictly positive value only if consumption or getting a coconut takes place.
- Notice also that we do not yet integrate over possible values of c.

- The second equality follows from considering a small time interval $\Delta\tau$.
- During it the agents changes the state with probability $\Delta\tau b(e(\tau))$.
- As a result the agent consumes and get utility y and changes his/her state the value of which is given by $V_U(\tau)$.
- The magnitude $o(\tau)$ signifies the case that our agent meets several partners.
- With a complementary probability the agent does not meet a partner and his/her expected utility is the same expressed in the first equality expect that time has run $\Delta\tau$ forward and must be accordingly discounted.

- The last expression can be given in a more convenient form

$$V_E(\tau) = [\Delta\tau b(e(\tau)) + o(\Delta\tau)]\,[y + V_U(\tau)]$$

$$+ [1 - \Delta\tau b(e(\tau)) - o(\Delta\tau)]\,e^{-r\Delta\tau}V_E(\tau + \Delta\tau)$$

- Our aim is to derive a differential equation for the value function, and for that purpose we express the above chain of equations as

$$V_E(\tau) - V_E(\tau + \Delta\tau) =$$

$$[\Delta\tau b(e(\tau)) + o(\Delta\tau)]\left[y + V_U(t) - e^{-r\Delta\tau}V_E(\tau + \Delta\tau)\right]$$

$$+ e^{-r\Delta\tau}V_E(\tau + \Delta\tau) - V_E(\tau + \Delta\tau)$$

- Divide everything by $\Delta\tau$ and take a limit to get the following expression

$$lim_{\Delta\tau\to 0}\frac{V_E(\tau)-V_E(\tau+\Delta\tau)}{\Delta\tau} =$$

$$lim_{\Delta\tau\to 0}\left[b(e(\tau))+\frac{o(\Delta\tau)}{\Delta\tau}\right]\left[y+V_U(\tau)-e^{-r\Delta\tau}V_E(\tau+\Delta\tau)\right]$$

$$+lim_{\Delta\tau\to 0}\frac{e^{-r\Delta\tau}-1}{\Delta\tau}V_E(\tau+\Delta\tau)$$

- After the limiting process we get

$$-\frac{dV_E(\tau)}{d\tau} = b(e(\tau))\left[y+V_U(\tau)-V_E(\tau)\right]-rV_E(\tau)$$

where we have assumed that the value function is continuously differentiable.

- The last term can be derived by the L'Hospital's rule:

$$lim_{\Delta t \to 0} \frac{e^{-r\Delta t} - 1}{\Delta t} = \frac{\frac{d}{dt}(e^{-rt} - 1)}{\frac{d}{dt} t} |_{t=0} = \frac{-re^{-rt}}{1} |_{t=0} = -r$$

  or by using the approximation $e^{-r\Delta t} = 1 - r\Delta t$.
- In a steady state the time derivative of the value function is zero and we get the standard asset value equation

$$rV_E = b(e) [y + V_U - V_E] \tag{1}$$

- Notice that focussing on a steady state means that none of the variables depends on time.

- The other value function can be derived analogously once one understands that the optimal strategy of collecting coconuts is a treshold strategy such that all coconut with costs $c \leq c^*$ are collected and the rest are not.

$$rV_U = a \int_{\underline{c}}^{c^*} [-c + V_E - V_U] g(c) dc \tag{2}$$

- We still have to figure out the treshold.
- It is clear that it must be given by the indifference condition

$$c^* = V_E - V_U \tag{3}$$

  because collecting a coconut amounts to changing the state from 'unemployment' to 'employment' and at the treshold an agent has to be indifferent.
- Notice that, once again, the value functions can be only implicitly determined.

- The final piece of the model to be determined is e, the measure of active agents.
- In a steady state the inflow of agents to e must equal the outflow at any point of time or

$$a(1-e)G(c^*) = b(e)e \tag{4}$$

where the LHS signifies that proportion a of the 'unemployed' find a coconut and proportion $G(c^*)$ of the coconuts is acceptable and the RHS signifies that of the 'employed' proportion $b(e)$ finds a partner.
- Since the determination of e involves $b(e)$ there is a possibility of multiple solutions.

- Solving the value functions (1) and (2), and inserting to (3) one finds the threshold

$$c^* = \frac{b(e)y + a\int_{\underline{c}}^{c^*} cg(c)dc}{r + b(e) + aG(c^*)} \tag{5}$$

and from (4) one can 'solve'

$$e = \frac{aG(c^*)}{aG(c^*) + b(e)} \tag{6}$$

- Equations (5) and (6) implicitly define the steady state values of the treshold and the measure of traders. As far as these two relationships hold simultaneously there exists an equilibrium. The problem is that in $c - e$-space both curves are upward sloping and there may be a multiplicity of equilibria.
- Expression (5) is equivalent to

$$c^* \left[r + b(e) + aG\left(c^*\right)\right] - b(e)y - a\int_{\underline{c}}^{c^*} cg(c)dc = 0$$

- Totally differentiating it one gets

$$dc^* \left\{r + b(e) + aG\left(c^*\right) + c^* ag\left(c^*\right) - c^* ag\left(c^*\right)\right\}$$

$$+de\left\{c^* b'(e) - b'(e)y\right\} = 0$$

from which one gets

$$\frac{dc^*}{de} = \frac{b'(e)\,(y - c^*)}{r + b(e) + aG\,(c^*)} > 0$$

- Analogously one gets from (6)

$$\frac{de^*}{dc} = \frac{a(1-e)g\,(c^*)}{b(e) + eb'(e) + aG\,(c^*)} > 0$$

- There are several things to notice.

- First, $e = 0$ always constitutes an equilibrium with $c^* = \underline{c}$; if no-one collects coconuts then no-one will trade them and then it is optimal not to collect them, and $e = 0$.
- Second, in all equilibria there are too few agents searching or e is too low because c^* is too low.
- This can be shown formally although it is quite cumbersome.
- The heuristics are clear though; an individual decision to increase c^* causes a positive external effect as there will be more traders in the economy.
- Individual decision making does not take this effect into account.

Example

- All production costs are the same c.
- Function $b(e) = \beta e$ where $\beta > 0$.
- We determine the Bellman equations, e and the probability that a production opportunity is accepted γ.
- The asset value equations are given by (notice the bad practice of 'e' denoting two different things)

$$rV_e = \beta e(y + V_u - V_e)$$

$$rV_u = amax_\gamma \gamma (V_e - V_u - c)$$

- Flow condition for employment in a steady state is given by

$$\beta ee = a\gamma(1-e)$$

from which we get

$$\gamma = \frac{\beta e^2}{a(1-e)}$$

- There are three possible cases: $\gamma = 0$, $0 < \gamma < 1$ or $\gamma = 1$.
- It is clear that the first one is always a solution: If no one produces then it does not pay to produce because eating requires exchange.
- If $\gamma = 1$ it must be the case that $V_e - V_u \geq c$.
- This is equivalent to $e \geq \frac{rc}{\beta(y-c)}$ (make sure you can derive this).
- If the inequality is to the other direction we get the necessary condition for $\gamma = 0$.

- More interesting is when $0 < \gamma < 1$.
- Then $V_e - V_u = c$.
- Then it is immediate that $V_u = 0$, and $V_e - V_u = \frac{\beta e y}{r + \beta e}$.
- Analogously to the previous cases we also have $e = \frac{rc}{\beta(y - c)}$.

- In (γ, e)-space we can see that there are three steady states.
- First is $(0,0)$.
- The second is $\left(\frac{\beta e^2}{a(1-e)}, \frac{rc}{\beta(y-c)}\right)$.
- And the third one is $\left(1, \frac{-a+\sqrt{a^2+4a\beta}}{2\beta}\right)$ where the value of e is got from the expression for γ which is unity here.
- These steady states can be Pareto-ranked so that the higher ones are better than the lower ones.