Mortensen-Pissarides model

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- Two-sided search model with prices determined by bargaining.
- There is a continuum of workers, say, a unit interval.
- There is also large mass, perhaps to be specified later, of firms.
- Agents discount future by factor $\delta = \frac{1}{1+r}$.

- Consumption at time t is given by c_t, and the workers' preferences at time τ by

$$E_\tau \sum_{t=\tau}^{\infty} \delta^{t-\tau} c_t$$

- The firms' preferences are given by

$$E_\tau \sum_{t=\tau}^{\infty} \delta^{t-\tau} (\pi_t - x_t)$$

where π_t denotes a firm's profit and x_t costs of keeping a vacancy open in period t.

- Goods are perishable and cannot be saved.
- Consequently, there are no savings in the economy and the workers eat all they earn in each period.
- The meetings of workers and firms are not explicitly modelled.
- Instead, a black-box matching function is used.
- If the mass of unemployed is u and the mass of vacancies is v then $m(u, v)$ matches are formed within a period.
- One has to require m to possess some properties.
- Usually the following are assumed:

1. m is continuous
2. m is increasing in both arguments
3. $m(0, v) = m(u, 0) = 0$ for all $u, v \geq 0$.
4. m is homogeneous of degree 1.

- The last assumption means that $m(\lambda u, \lambda v) = \lambda m(u, v)$ for $\lambda > 0$.

- The probability of an unemployed to be matched with a vacancy is given by $\frac{m(u,v)}{u} = m\left(1, \frac{v}{u}\right)$.
- The probability that a vacancy is matched with an unemployed is given by $\frac{m(u,v)}{v} = m\left(\frac{u}{v}, 1\right)$.
- Many times it is useful to parameterise the setting by labour market tightness $\theta = \frac{v}{u}$.
- For the rest we assume that $lim_{\theta \longrightarrow 0} m\left(\frac{1}{\theta}, 1\right) = lim_{\theta \longrightarrow \infty} m(1, \theta) = 1$.

- This is a world where one firm employes at most one worker, all workers are identical as are all firms.
- A matched pair produces output y.
- Matched pair gets separated with probability s.
- During unemployment the workers receive benefit b.
- Keeping a vacancy open costs x per period for firms.
- The non-trivial part is to determine the wage that the worker gets.
- For this we apply Nash-bargaining solution.

- The interpretation is that a firm and a worker that are matched negotiate a wage.
- Technically the wage w is determined by the Nash-bargaining solution.
- For this we interpret the parties' expected life time utilities while they wait for a partner as their threat points or disagreement points.
- Let us denote them at this stage by U for the unemployed and V for the vacancy, and determine them later on.
- In a steady state they are just numbers from the point of view of the worker and the firm.
- If a worker is employed at wage w his/her expected life time utility is given by $U(w)$ and that of the firm by $V(w)$.

- Then the Nash-bargaining solution is given by the solution to

$$max_w \left(U(w) - U\right)^{\gamma}\left(V(w) - V\right)^{1-\gamma}$$

subject to the individual rationality constraints

$$U(w) \geq U$$

$$V(w) \geq V$$

- As usual we ignore the constraints, determine the unconstrained optimum and make certain later on that the constraints are satisfied.

- The first order condition is given by

$$\gamma(U(w) - U)^{\gamma-1}\, U'(w)(V(w) - V)^{1-\gamma} + (U(w) - U)^{\gamma}(1-\gamma)(V(w) -$$

which is equivalent to

$$\gamma U'(w)(V(w) - V) + (1-\gamma)V'(w)(U(w) - U) = 0$$

- Let us next determine the value functions which are evaluated at the end of a period. A worker with wage w expects life time utility

$$U(w) = \delta\,(w + (1-s)U(w) + sU)$$

and a firm that pays wage w expects life time profits

$$V(w) = \delta\,(y - w + (1-s)V(w) + sV)$$

- We manipulate these equations a little to get the following equivalent asset value formulations

$$rU(w) = w + s(U - U(w)) \tag{1}$$

$$rV(w) = y - w + s(V - V(w)) \tag{2}$$

Solving (1) and (2) yields

$$U(w) = \frac{w + sU}{r + s} \tag{3}$$

$$V(w) = \frac{y - w + sV}{r + s} \tag{4}$$

from which we immediately see that the derivatives are given by $U'(w) = -V'(w) = \frac{1}{r+s}$.

- Using this, the relation that determines the wage is given by

$$\gamma(V(w)-V)-(1-\gamma)(U(w)-U)=0 \tag{5}$$

- Let us denote the unique wage determined by (5) by $\tilde{w}$.
- For equilibrium one needs the life time utilities as well as the equilibrium stocks of unemployed and vacancies.
- We already know that in a steady state equilibrium all firms pay the same wage and the value of searching is constant across agents.
- In a, by now, standard manner we can express the searching agents' value functions as

$$rU = b + m(1,\theta)(U(\tilde{w})-U) \tag{6}$$

$$rV = -x + m\left(\frac{1}{\theta},1\right)(V(\tilde{w})-V) \tag{7}$$

- Notice that we did not specify the number/measure of firms, and the reason for that is the vacancy cost.
- Presumably no firm would stay in the market if it makes negative profits, and if firms make positive profits more firms are attracted to the market.
- Thus, we require that the firms make zero profit in equilibrium $V = 0$, and this condition determines the measure of active firms.
- Inserting the zero-profit condition into (7) and solving for $V(\tilde{w}) = \frac{x}{m\left(\frac{1}{\theta},1\right)}$ and doing the same thing for (4), and then solving from these two one finds that

$$\tilde{w} = y - \frac{s+r}{m\left(\frac{1}{\theta},1\right)}x$$

Solving U from (6) and inserting this into (3) one can solve

$$U(\tilde{w}) = \frac{\tilde{w}(r+m(1,\theta))+sb}{r(r+m(1,\theta)+s)}$$

- Inserting these data into (5) one gets another expression for

$$\tilde{w} = \frac{\gamma(r + m(1,\theta) + s)y + (1-\gamma)(r+s)b}{r + \gamma m(1,\theta) + s}$$

- Equating the above expressions for $\tilde{w}$ one finds the relationship that determines the market tightness θ or the measure of the firms in equilibrium.
- This is a condition in implicit form and consequently aiming at explicit expressions for the value functions is not very useful.
- Many times the most fruitful approach is to bypass the expressions for the agents' value functions and to focus on the so called surplus function.

- Define the surplus by

$$S = U(\tilde{w}) + V(\tilde{w}) - U - V$$

and remember that in our special case with free entry $V = 0$.

- Now we see from (5) that $U(\tilde{w}) - U = \gamma S$ or in Nash-bargaining the worker gets share γ of the surplus.
- Analogously the firm gets share $1 - \gamma$ of the surplus or V($\tilde{w}$)-V=(1-γ)S.
- Manipulating (1), (2) and (6) one gets

$$r(U(\tilde{w}) + V(\tilde{w}) - U) =$$

$$y - b - s(U(\tilde{w}) + V(\tilde{w}) - U) - m(1, \theta)(U(\tilde{w}) - U)$$

- All the terms in parenthesis can be expressed in terms of S and we get

$$S = \frac{y-b}{r+s+\gamma m(1,\theta)} \tag{8}$$

- Using (7) we also find that

$$S = \frac{x}{(1-\gamma)m\left(\frac{1}{\theta},1\right)} \tag{9}$$

- These two equations allow the determination of the zero-profit level of firms or the equilibrium value of θ as well as that of S.
- Given S, from (2) one finds the equilibrium wage

$$\tilde{w} = y-(r+s)(1-\gamma)S$$

the value functions

$$U(\tilde{w}) = \frac{\tilde{w}+s\gamma S}{r}$$

and

$$U = \frac{\tilde{w}+(s-r)\gamma S}{r}$$

- The steady state flows of workers into employment and out of employment have to equal or $um(1,\theta) = (1-u)s$ from which we get

$$u = \frac{s}{s + m(1,\theta)} \tag{10}$$

which means that

$$v = \frac{s\theta}{s + m(1,\theta)} \tag{11}$$

- Now we have all the elements to determine whether an equilibrium exists.
- They key relations are (8) and (9), and we want to figure out whether there exist θ, or the measure of firms, such that both are satisfied.
- When $\theta = 0$ the RHS of (8) is $\frac{y-b}{r+s}$ and it is decreasing in θreaching value $\frac{y-b}{r+s+\gamma}$ when θ grows indefinitely.
- When $\theta = 0$ the RHS of (9) is $\frac{x}{1-\gamma}$ and it is increasing in θ growing indefinitely when θ grows indefinitely.

- Consequently, an equilibrium exists if

$$\frac{y-b}{r+s} > \frac{x}{1-\gamma}$$

- Clearly, this relation holds if the unemployment benefit is not too large or the cost of opening a vacancy is not too large.
- In terms of vacancy cost what is needed is

$$x < \frac{(1-\gamma)(y-b)}{r+s}$$

- Using (8) and (9) and the relevant value at $\theta = 0$ and $\theta = \infty$ one finds that in equilibrium

$$\frac{y-b}{r+s+\gamma} < S < \frac{y-b}{r+s}$$

- One still has to make certain that the individual rationality constraints in the Nash-bargaining solution hold.
- Notice that equilibrium consists of a three-tuple $(u, \theta, \tilde{w})$ such that (10) is satisfied, from (7) under zero-profit condition $V(\tilde{w}) = \frac{x}{m\left(\frac{1}{\theta}, 1\right)}$ and wage satisfies equation $\tilde{w} = \frac{\gamma(r+m(1,\theta)+s)y+(1-\gamma)(r+s)b}{r+\gamma m(1,\theta)+s}$.

- One can perform comparative statics analysis but one has to realise that the derivatives refer to how one steady state equilibrium value changes into another steady state equilibrium value; the derivatives do not tell what happens during the change.
- Differentiating (8) with respect to y one gets $\frac{\partial S}{\partial y} = \frac{1}{r+s+\gamma m(1,\theta)}$.
- But as the surplus increases there will be more firms in the market.
- Differentiating (11) with respect to θ gives $\frac{\partial v}{\partial \theta} = \frac{s(s+m(1,\theta)-\theta m_2(1,\theta))}{(s+m(1,\theta))^2} = \frac{s(s+m_1(1,\theta))}{(s+m(1,\theta))^2}$ where we have used the homogeneity of degree one of the matching function.[1]

[1] $m(tu,tv) \equiv tm(u,v)$or equivalently $\frac{m(t,t\theta)}{u} \equiv \frac{tm(1,\theta)}{u}$. Now differentiate both sides with respect to t to get $\frac{m_1(t,t\theta)}{u} + \frac{m_2(t,t\theta)\theta}{u} = \frac{m(1,\theta)}{u}$. Evaluate this at $t=1$ to get $\frac{m_1(1,\theta)}{u} + \frac{m_2(1,\theta)\theta}{u} = \frac{m(1,\theta)}{u}$.

- Looking, at (10) one immediately notices that vacancies and unemployment are negatively associated.
- This relation is called the Beveridge-curve.
- Let us manipulate the above relations to a form that is more amenable to analysis.
- First notice that from (5) $U(\tilde{w}) - U = \gamma S$.
- Going a little backwards express the surplus in terms of value functions

$$U(\tilde{w}) - U = \gamma(U(\tilde{w}) + V(\tilde{w}) - U - V) \qquad (12)$$

- Inserting expressions (3) and (4) in this and taking into account that $V = 0$ we get

$$\tilde{w} = rU + \gamma(y - rU) \qquad (13)$$

- For empirical applications rU is not an interesting variable.
- Consequently, in (12) insert the zero profit condition $V(\tilde{w}) = \frac{x}{m\left(\frac{1}{\theta},1\right)}$ and solve

$$U(\tilde{w}) - U = \frac{\gamma x}{(1-\gamma)m\left(\frac{1}{\theta},1\right)} \tag{14}$$

- Insert this into (6) to get

$$rU = b + \frac{\gamma}{1-\gamma} x\theta \tag{15}$$

where we have utilised the fact that $\frac{m(1,\theta)}{m\left(\frac{1}{\theta},1\right)} = \frac{m(1,\theta)}{\frac{1}{\theta}m(1,\theta)} = \theta$.

- Plug in (14) into (13) to get

$$\tilde{w} = (1-\gamma)b + \gamma(x\theta + y) \tag{16}$$

- Expression (16) replaces the labour supply curve of Walrasian markets.
- Even though supply is fixed in $\theta - w$-space there is an upward sloping relationship called the *wage curve*.
- Notice from (24) that in equilibrium a job creates a positive surplus both for the worker and the firm.
- The firm's net return $V(\tilde{w})$ is equal to the expected hiring costs.
- For an equilibrium to exist the production y must be greater than the worker's wage.

- Let us gather the relations that define the equilibrium

$$u = \frac{s}{s + m(1, \theta)}$$

$$\tilde{w} = y - \frac{s + r}{m\left(\frac{1}{\theta}, 1\right)} x$$

$$\tilde{w} = (1 - \gamma) b + \gamma (x\theta + y)$$

- The second of these curves is called the job creation curve.
- It slopes downward in $\theta - w$-space.
- The wage curve slopes upward in the same space, and their intersection determines the equilibrium value of $\theta = \theta^*$.

- In the $u - v$-space the first equation determines so called Beveridge-curve that plots combinations of unemployed and vacancies that is consistent with the steady state.
- Drawing in the same space the line with slope θ^* plots the combinations of u and v that are consistent with θ^*.
- The intersection of these curves determines the equilibrium u and v.
- Using the figures one can study what happens when the parameters of the model change.
- One immediately sees from the job creation curve that increase in productivity shifts it up, increasing the cost of vacancy, the discount rate or the separation rate rotates it down, while increasing the matching efficiency rotates it up.
- The first three effects increase the value of employing a worker which increases vacancy creation at a fixed wage.
- The fourth effect goes to the opposite direction.

- The wage curve shifts up when productivity or unemployment benefit increase.
- It rotates up when the bargaining power of the workers increases.
- Higher productivity increases the surplus from a match and consequently the wage at a fixed θ.
- Higher unemployment benefit increases the value of unemployment or the threat point of the worker in Nash-bargaining.
- This leads to higher wage at a fixed θ.
- Higher bargaining power increases the worker's share of the surplus, i.e., his/her wage at a fixed θ.

- To find the overall effect of a parameter change on typically first determines its effect on the wage curve, and the job creation curve.
- This determines new values for the wage and the job market tightness θ.
- Then one determines the impact of the new θ, and the resulting new labour tightness line, on the Beveridge curve.
- If one wants to track the parameter changes in r and s on both curves one can substitute to the wage curve $x\theta = (1-\gamma)m(1,\theta)\frac{y-b}{r+\gamma m(1,\theta)+s}$.
- This can be derived by using the two expressions for $\tilde{w}$.
- With this the wage curve becomes

$$\tilde{w} = b + \gamma(y-b)\frac{r+m(1,\theta)+s}{r+\gamma m(1,\theta)+s}$$

- When both the wage curve and the job creation curve move to the same direction the overall effect of parameter changes has to be determined non-graphically.
- Equate the RHS of each equation to get a new function

$$Z = y - b - \gamma(y-b)\frac{r + m(1,\theta) + s}{r + \gamma m(1,\theta) + s} - \frac{s + r}{m\left(\frac{1}{\theta}, 1\right)}x = 0$$

- Totally differentiating this expression we can determine the effect of any parameter change to θ. Let $\alpha \in \{y, b, r, s, x, \gamma\}$.
- Then

$$\frac{\partial \theta}{\partial \alpha} = -\frac{Z_\alpha}{Z_\theta}$$

- It is immediate that $Z_\theta < 0$.
- As an example it is straigthforward to show that $Z_y > 0$ and consequently increase in productivity increases labour market tightness θ.
- The change in wage can then be obtained from the wage curve.
- Now in the $u - v$-space with the Beveridge curve the labour tightness line increases in slope and consequently there will be more vacancies and less unemployed.