Trading mechanisms

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- S sellers each with unit supply of an indivisible good; value it at zero.
- B buyers each with unit demand; value a good at unity.
- The buyers contact the sellers, and then trade.
- The key questions are how are prices decided and how do the contacts happen.
- We study three price formation mechanisms: Auction, bargaining and posted prices.
- The economy proceeds in discrete time; discounting with factor $\delta \in (0, 1)$.
- Agents who trade are replaced by identical agents.

Auction

- Buyers contact the sellers randomly; each seller with the same probability.
- When there are very many buyers and sellers a seller expects k buyers with probability $e^{-\theta}\frac{\theta^k}{k!}$, $\theta = \frac{B}{S}$.
- If a solitary buyer meets a seller s/he offers a price that just makes the seller indifferent.
- If several buyers meet a seller they bid up the price until they are indifferent between trading and searching for a new partner.
- Focus on steady state equilibrium.
- Evaluate buyers' and sellers' value functions, W and Y, at the end of a period.

- We need not figure out actual prices: If only one buyer contacts a seller the buyer makes an offer that leaves the seller indifferent between accepting and rejecting i.e., waiting till next period.
- If there are two or more buyers competing for a good then the price will be so high that the buyers are indifferent between buying at the price and waiting till next period.
- Now the value functions are given by

$$W = \delta \left\{ e^{-\theta}(1-Y) + \left(1-e^{-\theta}\right) W \right\}$$

$$Y = \delta \left\{ e^{-\theta} Y + \theta e^{-\theta} Y + \left(1-e^{-\theta} - \theta e^{-\theta}\right)(1-W) \right\}$$

- Solving the equations explicitly is straightforward

$$W = \delta \frac{e^{-\theta}}{1-\delta\theta e^{-\theta}}$$

$$Y = \delta \frac{1-e^{-\theta}-\theta e^{-\theta}}{1-\delta\theta e^{-\theta}}$$

Bargaining

- The seller chooses randomly one of the buyers who contacts him/her and they split the available surplus in half.
- Denote a buyer's value function by U and that of a seller by V.
- Notice that a symmetric contact strategy means that each seller is contacted with the same probability or the expected queue length at each seller is θ.
- When a buyer and seller negotiate about the price the end result is the division of the surplus $1 - U - V$ in half.
- The probability that a buyer is selected as a bargaining partner is given by

$$\sum_{i=0}^{\infty} e^{-\theta} \frac{\theta^i}{i!} \frac{1}{i+1} = \frac{1}{\theta} \sum_{i=1}^{\infty} e^{-\theta} \frac{\theta^i}{i!} = \frac{1-e^{-\theta}}{\theta}$$

- A seller finds a bargaining partner with probability $1-e^{-\theta}$.
- The value functions look as follows

$$U=\delta\left\{\frac{1-e^{-\theta}}{\theta}\left(U+\frac{1}{2}(1-U-V\right)+\frac{\theta-1+e^{-\theta}}{\theta}U\right\}$$

$$V=\delta\left\{\left(1-e^{-\theta}\right)\left(V+\frac{1}{2}(1-U-V\right)+e^{-\theta}V\right\}$$

- Solving these equations explicitly

$$U=\delta\frac{1-e^{-\theta}}{\theta\left(2-\delta-\delta e^{-\theta}\right)+\delta(1-e^{-\theta})}$$

$$V=\delta\frac{\theta\left(1-e^{-\theta}\right)}{\theta\left(2-\delta-\delta e^{-\theta}\right)+\delta(1-e^{-\theta})}$$

Evolutionary stability

- One would perhaps like to compare the above trading mechanisms.
- An obvious criterion would be efficiency but it has no bite as equal number of trades takes place under either scenario.
- One way is to consider disequilibrium dynamics.
- To that end assume that there are two market places; in one trades are consummated by bargaining and in the other by auction.
- Let proportion x of the buyers and proportion y of the sellers participate in the bargaining market and the rest in the auction market.
- Now the formulas of the life time utilities remain of the same form but the expected queue lengths change to $\alpha = \frac{x}{y}\theta$ and $\beta = \frac{1-x}{1-y}\theta$.
- If there is an equilibrium where the two markets exist simultaneously then the buyers have to fare equally well in both market, and sellers, too.

- Formally, this means that $U = W$ and $V = Y$ or

$$\delta \frac{1-e^{-\alpha}}{\alpha(2-\delta-\delta e^{-\alpha})+\delta(1-e^{-\alpha})} = \delta \frac{e^{-\beta}}{1-\delta\beta e^{-\beta}}$$

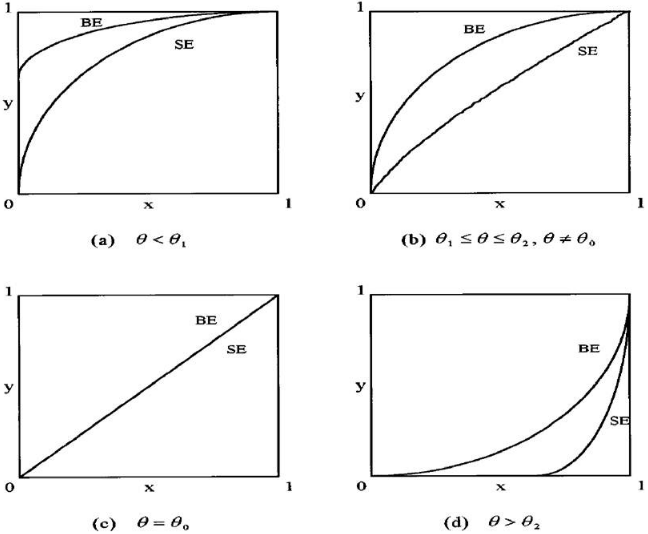
$$\delta \frac{\alpha(1-e^{-\alpha})}{\alpha(2-\delta-\delta e^{-\alpha})+\delta(1-e^{-\alpha})} = \delta \frac{1-e^{-\beta}-\beta e^{-\beta}}{1-\delta\beta e^{-\beta}}$$

- These are equivalent with

$$\frac{1-e^{-\alpha}}{\alpha(2-\delta-\delta e^{-\alpha})+\delta(1-e^{-\alpha})}-\frac{e^{-\beta}}{1-\delta\beta e^{-\beta}}=0$$

$$\frac{\alpha(1-e^{-\alpha})}{\alpha(2-\delta-\delta e^{-\alpha})+\delta(1-e^{-\alpha})}-\frac{1-e^{-\beta}-\beta e^{-\beta}}{1-\delta\beta e^{-\beta}}=0$$

  where the first one is called the buyers' equilibrium curve and the latter one the sellers' equilibrium curve.
- Both determine y as a function of x.
- Somewhat lengthy analysis that can be found in Lu and McAfee (1996 Games and Economic Behavior, 15, 228-254) shows that the curves are as in the following figure.
- Notice that naturally the shape of the curves depends on the overall ratio of buyers to sellers.


1
BE
SE
y
0
x
1
(a) θ < θ1
1
BE
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(b) θ1 ≤ θ ≤ θ2 , θ ≠ θ0
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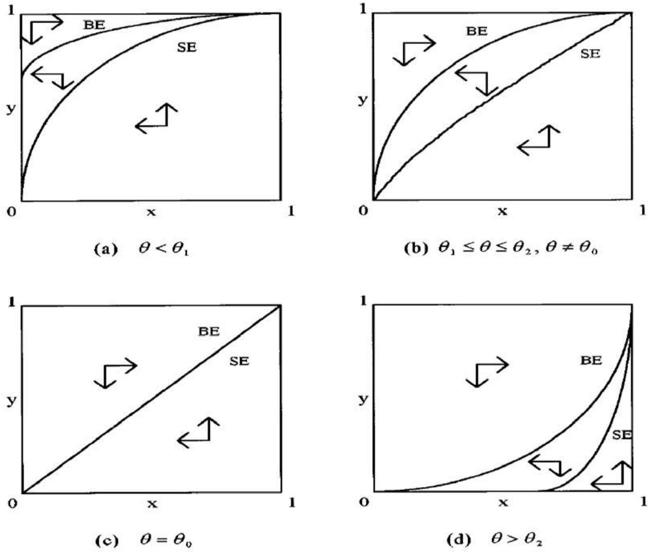
0
x
1
(c) θ = θ0
1
BE
SE
y
0
x
1
(d) θ > θ2

- First, one can see that the curves do not intersect at interior points which means that the two markets cannot coexist.
- Thus, there are two equilibria; either everyone consummates the trades by bargaining or everyone consummated the trades by auction.
- To better understand the relative performance of the two trading mechanisms one considers dynamics.
- In this case replicator dynamics turns out convenient. T
- The basic idea is that each period small proportion of entering agents adapt; they go to the market where their type did better in the previous period.
- Consequently, one can analyse what happens even in disequilibrium states.

- The idea of the dynamics is to find a stable equilibrium.
- An equilibrium is stable if small perturbations away from it do not upset it.
- If the economy is not in equilibrium but close to stable equilibrium then the dynamics goes towards this equilibrium.
- The figures show the direction of movement.
- It is almost immediate that only the auction market constitutes a stable equilibrium.
- The exception is the case when $\theta = \theta_0$ which is the value that makes BE- and SE-curves coincide on the diagonal.
- Then no equilibrium is stable.


1
BE
SE
y
0
x
1
(a) θ < θ₁
1
BE
SE
y
0
x
1
(b) θ₁ ≤ θ ≤ θ₂, θ ≠ θ₀
1
BE
SE
y
0
x
1
(c) θ = θ₀
1
BE
SE
y
0
x
1
(d) θ > θ₂

Posted price

- The sellers compete in prices which they announce to the buyers.
- Based on the prices the buyers contact the sellers.
- A seller with a lower price than his/her competitors can expect more buyers than his/her competitors.
- Then all the sellers do not experience the same queue lengths like in the analysis of bargaining and auction.
- On top of that it is unclear how to model price posting.
- The standard way would be to assume that some price is an equilibrium price, and then consider a single seller who deviates to different price, calculate the optimal deviation, and impose that it has to be the original price.
- But a single seller is of measure zero and it is difficult to come up with a story how the queue length that s/he expects is determined.

- Let us see how this problem can be solved.
- Focus on symmetric equilibrium and assume that the equilibrium price is p.
- Evaluate the sellers' and buyers' life-time utilities at the end of a period and denote them by S and B

$$S = \delta \left\{ e^{-\theta} S + \left(1 - e^{-\theta}\right) p \right\}$$

$$B = \delta \left\{ \sum_{k=0}^{\infty} e^{-\theta} \frac{\theta^k}{k!} \frac{1}{k+1} (1-p) + \left(1 - \sum_{k=0}^{\infty} e^{-\theta} \frac{\theta^k}{k!} \frac{1}{k+1} \right) B \right\}$$

$$= \delta \left\{ \frac{1 - e^{-\theta}}{\theta} (1-p) + \frac{\theta - 1 + e^{-\theta}}{\theta} B \right\}$$

- One can solve

$$S = \delta \frac{1 - e^{-\theta}}{1 - \delta e^{-\theta}} p$$

$$B = \delta \frac{1 - e^{-\theta}}{\theta(1 - \delta) + \delta (1 - e^{-\theta})} (1 - p)$$

- One still has to determine the equilibrium value of p.

- Assume that proportion z of sellers deviates for one period and posts price p'.
- Proportion σ of buyers contact the deviators and $1-\sigma$ the non-deviators.
- Meeting rate for deviating sellers $\theta' = \frac{\sigma}{z}\theta$ and for non-deviating sellers $\widetilde{\theta} = \frac{1-\sigma}{1-z}\theta$.
- These are determined by the buyers' indifference

$$\frac{1-e^{-\theta'}}{\theta'}(1-p') + \frac{\theta' - 1 + e^{-\theta'}}{\theta'}B$$

$$= \frac{1-e^{-\widetilde{\theta}}}{\widetilde{\theta}}(1-p) + \frac{\widetilde{\theta} - 1 + e^{-\widetilde{\theta}}}{\widetilde{\theta}}B$$

- For the deviating sellers' programme we need to know how σ depends on p'.
- Solving

$$p' = 1 + \frac{\theta' - 1 + e^{-\theta'}}{\theta'} B$$

$$- \frac{\theta'}{1 - e^{-\theta'}} \left\{ \frac{1 - e^{-\widetilde{\theta}}}{\widetilde{\theta}} (1 - p) + \frac{\widetilde{\theta} - 1 + e^{-\widetilde{\theta}}}{\widetilde{\theta}} B \right\}$$

- Insert this into the sellers' objective

$$max_{\sigma} e^{-\theta'} S + \left(1 - e^{-\theta'}\right) p'$$

and derive the first order condition, as $\frac{\partial \theta'}{\partial \sigma} = \frac{\theta}{z}$ and $\frac{\partial \widetilde{\theta}}{\partial \sigma} = -\frac{\theta}{1-z}$,

$$-e^{\theta'} S/z + e^{\theta'}/z + \left(1 - e^{\theta'}\right) \frac{\theta B}{z(1-z)\widetilde{\theta}^2} - e^{\theta'} \frac{\theta' B}{(1-z)\widetilde{\theta}} - e^{\theta'} \frac{B}{z}$$

$$+ \left[e^{\theta'} \frac{\theta'}{(1-z)\widetilde{\theta}} - \left(1 - e^{\theta'}\right) \frac{\theta}{z(1-z)\widetilde{\theta}^2} \right] (1-p) = 0$$

- Inserting the expressions of lifetime utilities and then imposing equilibrium condition $p' = p$, and then letting z go to zero one gets (a candidate) equilibrium price

$$p = \frac{\left(1 - \delta e^{-\theta}\right)\left(1 - e^{-\theta} - \theta e^{-\theta}\right)}{\left(1 - e^{-\theta}\right)\left(1 - \delta \theta e^{-\theta}\right)}$$

- This price is just a local maximum; for equilibrium one has to consider large deviations, too.
- We do this a little later.
- Inserting the equilibrium price into the expressions for life time utilities gives

$$B = \delta \frac{e^{-\theta}}{1 - \delta \theta e^{-\theta}}$$

$$S = \delta \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - \delta \theta e^{-\theta}}$$

- One immediately finds that $W = B$ and $Y = S$.
- This means that auction and posted price are equivalent trading mechanisms utilitywise.
- This is a pretty neat result but only applies with homogeneous buyers and sellers provided that they are risk neutral.

- Finally, let us show that the above p is an equilibrium price.
- Let us also adopt the so called competitive market technique where it is assumed that if someone deviates s/he experiences queue length that guarantees the buyers the same utility as they would get when no-one deviates, so called market utility.
- Assume that a seller deviates to price q.
- Buyers approach the deviator with rate γ such that

$$(1-q)\frac{1-e^{-\gamma}}{\gamma}+\frac{\gamma-1+e^{-\gamma}}{\gamma}B=(1-p)\frac{1-e^{-\theta}}{\theta}+\frac{\theta-1+e^{-\theta}}{\theta}B$$

- Solving for q one finds that the deviator expects profit

$$(1-e^{-\gamma})q+e^{-\gamma}S=$$

$$1-e^{-\gamma}+\left(\gamma-1+e^{-\gamma}\right)B-\gamma\frac{e^{-\theta}\left[\theta(1-\delta)+\delta\left(1-e^{-\theta}\right)\right]}{(1-e^{-\theta})(1-\delta\theta e^{-\theta})}\frac{\left(1-e^{-\theta}\right)}{\theta}-$$

$$\gamma\frac{\theta-1+e^{-\theta}}{\theta}\delta\frac{e^{-\theta}}{1-\delta\theta e^{-\theta}}+e^{-\gamma}S$$

which is equivalent to

$$1-e^{-\gamma}+\left(\gamma-1+e^{-\gamma}\right)B-$$

$$\gamma\frac{e^{-\theta}}{1-\delta\theta e^{-\theta}}+e^{-\gamma}S$$

- Let us study the magnitude of this expression, and denote it by $f(\gamma)$.
- Notice first that $f(0) = S > 0$ and $f(\infty) > 0$.
- The first derivative is $f^{'}(\gamma) = e^{-\gamma} + (1 - e^{-\gamma})\, B - \frac{e^{-\theta}}{1-\delta\theta e^{-\theta}} - e^{-\gamma}S$, and the second derivative is $f^{''}(\gamma) = -e^{-\gamma} + e^{-\gamma}B + e^{-\gamma}S < 0$.
- Thus, f is concave and the unique zero of its derivative gives its maximum value.

- Inserting $\gamma = \theta$ into the derivative yields

$$e^{-\theta} + \left(1 - e^{-\theta}\right) \delta \frac{e^{-\theta}}{1 - \delta \theta e^{-\theta}} - \frac{e^{-\theta}}{1 - \delta \theta e^{-\theta}}$$

$$-e^{-\theta} \delta \frac{1 - e^{-\theta} - \theta e^{-\theta}}{1 - \delta \theta e^{-\theta}} = 0$$

- This means that the optimal deviation is such that the expected queue length is θ; but there is just one price that generates that queue lenght namely p.