Solvalla School May 2018: Index Notation Exercises

1. In tensor notation with the Einstein summation convention the Lorentz transformation can be written as

$$x^\rho \to x'^\rho = \Lambda^\rho{}_\sigma \, x^\sigma$$

The space-time interval can be written

$$ds^2 = \eta_{\mu\nu} \, dx^\mu \, dx^\nu$$

Show that invariance of the interval under Lorentz transformations implies that

$$\eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$$

2. Write out in component form the vector equation

$$a'^\mu = M^\mu{}_\nu a^\nu$$

without making any assumption as to the form of $M^\mu{}_\nu$. If $M^\mu{}_\nu = \delta^\mu_\nu$, the Kronecker δ, show that $a'^\mu = a^\mu$.

3. Show that $\eta_{\mu\nu}\eta^{\mu\nu} = 4$.

4. The inverse Lorentz transformation is represented as $(\Lambda^{-1})^\mu{}_\nu$, defined so that

$$(\Lambda^{-1})^\mu{}_\nu \, \Lambda^\nu{}_\rho = \delta^\mu_\rho$$

Considering the special case of a Lorentz boost with speed v in the x^1-direction, verify that the inverse is obtained by substituting $v \to -v$.

5. State whether the following are satisfactory equations consistent with Special Relativity. If they are not state why. If they are, state the tensor rank of the equation.

a) $\partial_\mu a_\nu = M_{\mu\rho}$ b) $a_\mu a^\mu = \partial_\nu a^\nu$ c) $\partial_\rho M^{\rho\sigma} N_\sigma{}^\nu P^\rho{}_\lambda = \partial^\nu A$ d) $\dfrac{\partial u^\mu}{\partial \tau} = eF^{\mu\nu} u_\nu$

6. Find the transformation law of $\frac{\partial}{\partial x_\mu}$. Is it consistent to write this as ∂^μ, with $\partial^\mu = \eta^{\mu\nu}\partial_\nu$, given the transformation law for ∂_μ?