Special Relativity and Index Notation

Mark Hindmarsh
Solvalla May 2018

Coordinate 4-vector Put space and time coordinates of an event together into a space-time coordinate $x^\mu = (ct, x, y, z)$. Greek indices e.g. μ, ν, ρ take values $0, 1, 2, 3$. Use Roman indices e.g. i, j, k taking values $1, 2, 3$ to denote spatial components: e.g. $x^i = (x, y, z)$ with Cartesian coordinates.

Space-time interval In infinitesimal form ds, with $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. Lorentz invariant or *scalar*.

Minkowski metric Rewrite interval as $ds^2 = \sum_{\mu,\nu} dx^\mu \eta_{\mu\nu} dx^\nu$., with

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

(Here, μ labels rows and ν columns.)

Summation convention Sum over repeated indices e.g. $ds^2 = dx^\mu \eta_{\mu\nu} dx^\nu$.

Kronecker $\boldsymbol{\delta}$ A unit matrix in index notation:

$$\delta^\mu_\nu = \begin{cases} 1 & \text{if} \quad \mu = \nu, \\ 0 & \text{if} \quad \mu \neq \nu. \end{cases}$$

Inverse metric Defined by

$$\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_\rho.$$

Lorentz transformation Coordinates x^μ transform as $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$. LT to frame moving with speed v in the $+x$ direction:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma v/c & \cdot & \cdot \\ -\gamma v/c & \gamma & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

Contravariant 4-vector Transforms in the same way as the coordinate 4-vector:

$$a^\mu \to a'^\mu = \Lambda^\mu{}_\nu a^\nu$$

Example: 4-momentum $p^\mu = (E/c, p^i)$.

Inverse Lorentz transformation Defined by

$$(\Lambda^{-1})^\mu{}_\nu \Lambda^\nu{}_\rho = \delta^\mu_\rho,$$

so

$$a^\mu = (\Lambda^{-1})^\mu{}_\nu a'^\nu.$$

Example inverse LT back from frame moving with speed v in $+x$ direction:

$$(\Lambda^{-1})^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma v/c & \cdot & \cdot \\ \gamma v/c & \gamma & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

4-vector differentiation Transforms with *inverse*: $\frac{\partial}{\partial x^\mu} \to \frac{\partial}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu \frac{\partial}{\partial x^\nu}$.

Covariant 4-vector By definition transforms as

$$a_\mu \to a'{}_\mu = (\Lambda^{-1})^\nu{}_\mu a_\nu$$

The partial derivative is a covariant 4-vector. *Convention:* we write with the index down, e.g. partial derivative is written ∂_μ.

Raising and lowering indices Multiplication by the metric converts between contra- and covariant.

$$b_\mu = \eta_{\mu\nu} b^\nu, \qquad b^\mu = \eta^{\mu\nu} b_\nu$$

Scalar product For any two 4-vectors a^μ, b^μ:

$$a \cdot b \equiv a_\mu b^\mu = a^\mu b_\mu.$$

One also often writes a^2 for $a \cdot a$. It is usually clear from the context that a is a 4-vector.

4-velocity Define *proper time* along a worldline by $d\tau = ds/c$. It is the time kept by a clock moving with the object. (Cannot do this if object is moving at c). 4-velocity is then

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma(c, v^i),$$

where $v^i = dx^i/dt$ is the measured velocity, or 3-velocity.

4-acceleration Defined by differentiating the 4-velocity again:

$$a^\mu = \frac{d^2x^\mu}{d\tau^2}.$$

Invariant d'Alembertian or wave operator

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Top Tips and Handy Hints for 4-vector algebra

1. With repeated indices, there is usually one covariant and one contravariant index: this ensures that the equation has a definite property under Lorentz transformations (e.g. covariance or invariance). If the repeated index is not in such a pair, the expression has no definite property under Lorentz transformations.

2. Repeated indices are *dummy indices*. Dummy indices can always be relabeled to make the equation more convenient or clearer. For example, if I have the equation $a_\mu b^\mu = a_\rho c^\rho$, for all 4-vectors a, I can relabel the right hand side to get $a_\mu b^\mu = a_\mu c^\mu$, or $a_\mu(b^\mu - c^\mu) = 0$. Note that the solution to this equation is $b^\mu = c^\mu + q^\mu$, with $a \cdot q = 0$.

3. In order for an equation to be relativistically covariant, free indices must match in every term in an equation, in both number and position (covariant or contravariant). An equation in which the free indices do not match (e.g. $Y^\mu = a_\mu$) may be true but it is not covariant.

4. Always make sure, by relabelling if necessary, that all indices are labelled distinctly. For example,

$$\Lambda^\mu{}_\rho \Lambda^\mu{}_\sigma \eta_{\mu\mu} = \eta_{\rho\sigma}$$

is bad, because μ has been used twice as a dummy index.