Gravitational waves

Details

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General Theory of Relativity

Fundamental objects

- Spacetime, coordinates x^μ ($\mu = 0, 1, 2, 3$)
- Metric tensor $g_{\mu\nu}(x)$ gives space-time interval:

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu$$

- Tensors describing curvature, e.g. Einstein tensor: $G_{\mu\nu}$
- Properties of Einstein tensor
  - Two derivatives of metric $g_{\mu\nu}(x)$:

$$G \sim g^{-1}\partial^2 g, \quad (g^{-1}\partial g)^2 .$$

  - Symmetric: $G_{\mu\nu} = G_{\nu\mu}$.
  - Vanishes in Minkowski spacetime: $G_{\mu\nu} = 0$
- Energy-Momentum tensor $T^{\mu\nu}$ (= $T^{\nu\mu}$, symmetric).

T^{00} energy density
T^{0i} energy flux
T^{ii} pressure
T^{ij} ($i \neq j$)shear stress

Einstein equation

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

- Matter curves space-time (and space-time tells matter how to move)
- Compatibility with Newtonian gravity gives $\kappa = 8\pi G/c^4$
  - Poisson equation, $\nabla^2\Phi = 4\pi G\rho$ (ρ - mass density, contained in T_{00})

Weak gravitational fields

- Relevant at large distances from objects
- Close to Minkowski space-time: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, $\forall\ \mu, \nu$
- Special Relativity holds approximately
- h is called the *metric perturbation*.
- Substitute into Einstein equation
  - Define *trace* of $h = \eta^{\nu\mu} h_{\mu\nu} = h^{\nu}{}_{\nu}$
  - Define *trace reverse*: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$
- Can choose coordinates so $\partial_\mu \bar{h}^{\mu}{}_{\beta} = 0$ (“de Donder” or “Lorenz” gauge).
- Einstein equation becomes

$$\boxed{\Box \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}}$$

- Wave or d’Alembertian operator:

$$\begin{aligned} \Box &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\ &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \end{aligned}$$

Weak field solution: gravitational waves in empty space

- Empty space means no energy-momentum source:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\bar{h}_{\mu\nu}=0$$

- Plane wave solution:

$$\bar{h}_{\mu\nu}=A_{\mu\nu}e^{-i\omega t+i\mathbf{k}\cdot\mathbf{x}}+\text{c.c}\equiv A_{\mu\nu}e^{ik\cdot x}+A^*_{\mu\nu}e^{-ik\cdot x},$$

  - $A_{\mu\nu}$ a constant symmetric 4×4 matrix
  - $k^\mu=(\omega/c,\mathbf{k})$
  - $\omega^2=c^2|\mathbf{k}|^2$, $k^2\equiv k\cdot k=0$
  - $k^\mu A_{\mu\nu}=0$ (*transverse*)
- Further coordinate choice gives
  - $A^\nu{}_\nu=0$ (*traceless*)
  - $A_{0\nu}=0=A_{\nu 0}$ (zeroth row and column vanish)
- Example: plane wave in z direction, $k^\mu=(\omega/c,0,0,k)$:

$$A_{\mu\nu}=\begin{pmatrix}0&0&0&0\\0&A_{xx}&A_{xy}&0\\0&A_{xy}&-A_{xx}&0\\0&0&0&0\end{pmatrix},$$

Gravitational waves: polarisations

- Plane wave solution:

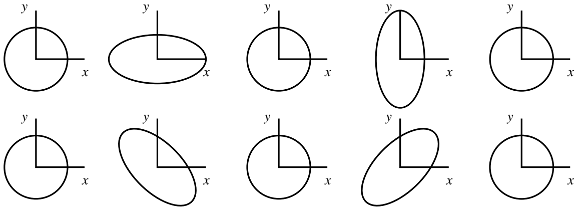
$$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik\cdot x} + A^*_{\mu\nu} e^{-ik\cdot x}$$

- Two free parameters: express in terms of basis matrices

$$\epsilon_+^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon_\times^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Hence, writing $h_+ = A_{xx}$, $h_\times = A_{xy}$

$$\bar{h}^{\mu\nu}(x) = \left(h_+ \epsilon_+^{\mu\nu} + h_\times \epsilon_\times^{\mu\nu}\right) e^{ik\cdot x} + \text{c.c.}\,.$$



Spherical wave

- Radial propagation outward from a source.
- Wave equation with spherical symmetry:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right)\bar{h}_{ij} = 0$$

- Solution

$$\bar{h}_{ij} = \frac{B_{ij}}{r}e^{-i\omega t + ikr}.$$

  with B a constant matrix.
- **Exercise: check that this is a solution of the radial wave equation, and that it describes outward propagation**

Generation of gravitational waves

- Spatial parts of Einstein equation:

$$\Box \bar{h}_{ij} = -2\kappa T_{ij}$$

- Gravitational wave is transverse and traceless; sourced by *shear stress* in energy-momentum tensor
- General solution ($t_{\rm r} = t - |\mathbf{x} - \mathbf{x}'|/c$ is *retarded time*):

$$\bar{h}_{ij}(t, \mathbf{x}) = -2\kappa \int d^3x' \, \frac{T_{ij}(t_{\rm r}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \, .$$

- For a distant source confined to a limited volume: **quadrupole formula**

$$\bar{h}^{ij}(t, r) = -\frac{2G}{r} \ddot{I\!\!\!-}^{ij}(t_{\rm r}) \, .$$

(Uses conservation of energy-momentum.)

  - $I\!\!\!-^{ij} = I^{ij} - \frac{1}{3} I^k_{\ k} \delta^{ij}$ is *quadupole moment*
  - $I^{kl} = \int d^3x \, x^k x^l \rho$ (second moment of mass density ρ)

Estimate of h

- Spherical object has no quadrupole moment.
- Equal mass stars mass m in circular orbit radius a

$$|\not{I}| \sim ma^2$$

- Second time derivative of

$$|\ddot{\not{I}}| \sim ma^2\omega^2$$

  with $\omega = 2\pi/P$, angular frequency of orbit period P.
- Hence (ignoring factors of 2 and π)

$$h \sim \frac{Gm}{c^2 r}\frac{a^2\omega^2}{c^2} = \frac{Gm}{c^2 r}\frac{v^2}{c^2}$$

  where $v = \omega a$ is orbital speed.
- **Exercise: estimate h caused by a pair of $30\,M_\odot$ black holes orbiting at $v/c = 0.5$ a distance 1000 Mly.**

- Gravitational wave energy density

$$e_{\rm gw} = \frac{c^2}{32\pi G}\langle \dot{\bar{h}}_{ij}\dot{\bar{h}}_{ij}\rangle$$

where $\langle\cdots\rangle$ denotes a time average over at least one period of the wave.
Exercise: Check this has the dimensions of an energy density.

- GW intensity

$$I_{\rm gw} = \frac{c^3}{32\pi G}\langle \dot{\bar{h}}_{ij}\dot{\bar{h}}_{ij}\rangle$$

Exercise: Check this has the dimensions of a power/area.

- GW luminosity of a source: integrate intensity over a large sphere

$$L_{\rm gw} = r^2\int d\Omega\,\frac{c^3}{32\pi G}\langle \dot{\bar{h}}_{ij}\dot{\bar{h}}_{ij}\rangle$$

- A difficult calculation[1] shows

$$L_{\rm gw} = \frac{1}{5}\frac{G}{c^5}\langle \dddot{I}_{ij}\dddot{I}_{ij}\rangle\,.$$

[1] Difficult because $\bar{h}_{ij}$ must be put into transverse traceless form in all directions.

Binary inspiral

- Binary masses m_1, m_2, total mass M.
- Circular orbit: CM distances a_1, a_2, separation $R = a_1 + a_2$.
- Moment of inertia $I_{\rm cm} = m_1 a_1^2 + m_2 a_2^2 = \mu R^2$ (*reduced mass* $\mu = m_1 m_2 / M$).
- Energy of system:

$$E = \frac{1}{2} I_{\rm cm}\omega^2 - \frac{G m_1 m_2}{R} = -\frac{1}{2} I_{\rm cm}\omega^2$$

- Kepler's Third Law:

$$\omega^2 R^3 = GM$$

- GW luminosity:

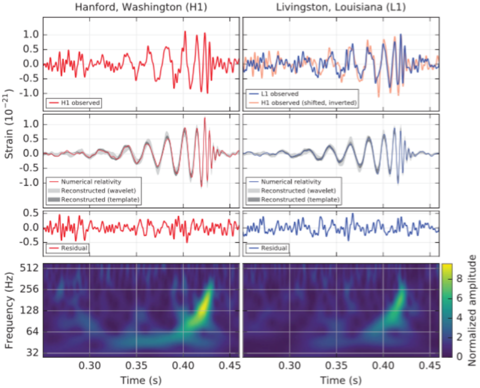
$$L_{\rm gw} = \frac{32}{5}\frac{G}{c^5} I_{\rm cm}^2 \omega^6 .$$

- Gravitational radiation results in loss of energy and decrease of period

$$\dot{E} = -L_{\rm gw} .$$

- Gravitational wave frequency f is twice the orbital frequency.

Exercise: assuming equal masses $m = m_1 = m_2$, derive the formula for the rate of change of the period P, and estimate the mass of the black holes in GW150914 from the data.
[Hint: use Kepler's law to eliminate the distance R from the equations]
It is useful to know that $2GM_\odot/c^2 \simeq 3$ km, where $M_\odot$ is the solar mass.



Further reading

- *The basic physics of the binary black hole merger GW150914*, LIGO Scientific and VIRGO Collaborations, Annalen der Physik **529** (2017) 1600209 [https://arxiv.org/abs/1608.01940]
- *'A First Course in General Relativity'*, B. Schutz, Cambridge University Press, 2009 (2nd ed.)
- *'Spacetime and Geometry'*, S. Carroll, Pearson, 2013
- *'Introducing General Relativity'*, M. Hindmarsh and A.R. Liddle, Wiley (to appear).