1. **Upper bound on the inflationary scale:** We do not now precisely the scale of inflation H_* but the upper limit on the tensor to scalar ratio $r < 0.09$ constrains it from above. See the lectures for the definition of r, use $\mathcal{P}_\zeta \simeq 2.1 \times 10^{-9}$ and find the maximal allowed H_*. Compute also the corresponding energy scale $\rho_*^{1/4}$.

2. **Horizon problem:** To explain the homogeneity of CMB the entire observable universe must originate from a single causal patch. Let us work out how many efolds of inflation are needed to arrange this. Consider the following simplified model for the history of universe:

$$\begin{aligned} t_{\rm in} < t < t_{\rm end} \quad &: \quad a = e^{H_* t} && \text{inflation} \\ t_{\rm end} < t < t_{\rm CMB} \quad &: \quad H = H_* \left(\frac{a_{\rm end}}{a}\right)^2 && \text{radiation domination} \\ t_{\rm CMB} < t < t_0 \quad &: \quad H = H_{\rm CMB} \left(\frac{a_{\rm CMB}}{a}\right)^{3/2} && \text{matter domination .} \end{aligned}$$

   Neglecting changes in the number of relativistic species, we have

$$T \propto a^{-1} \ .$$

   a) Compute the coordinate distance travelled by photons from an initial time $t_* < t_{\rm end}$ to $t_{\rm CMB}$ when CMB is formed

$$r_{\rm CMB}^{\rm hor} = \int_{t_*}^{t_{\rm CMB}} \frac{dt}{a(t)} \ .$$

   b) Compute the coordinate distance travelled by photons from $t_{\rm CMB}$ to today t_0

$$r_0^{\rm hor} = \int_{t_{\rm CMB}}^{t_0} \frac{dt}{a(t)} \ .$$

   c) To solve the horizon problem we need to have $r_{\rm CMB}^{\rm hor} > r_0^{\rm hor}$. Find the minimum number of efolds $N_* = \ln(a_{\rm end}/a_*)$ needed to satisfy this. Use that $T_{\rm CMB} = 0.3$ eV and $T_{\rm CMB}/T_0 = 1100$, and assume $T_{\rm end} = 10^{15}$GeV (which corresponds to $H_* \simeq 8 \times 10^{11}$GeV).

3. **Quadratic inflation:** Consider single field slow roll inflation with the inflaton potential given by

$$V(\phi) = \frac{1}{2} m^2 \phi^2 \ .$$

   a) Solve the slow roll equations of motion for $\phi(N)$ where N denotes the number of e-folds left until the end of inflation.

   b) Compute the power spectrum $\mathcal{P}_\zeta$ and the spectral index n_s as function of $\phi(N)$.

   c) The scales observable in the CMB exit the horizon at $N_* \simeq 60$. What is the inflaton value ϕ_* at this point?

   d) Find the mass scale m needed to give the observed spectrum $\mathcal{P}_\zeta \simeq 2.1 \times 10^{-9}$ for ϕ_*. How does the spectral index n_s compare to the observed value $n_s = 0.968 \pm 0.006$?