Numerics

Spectral method

The spectral method can be used to solve differential equations numerically. In this problem we will apply it to a simple 1-dimensional situation to see how it works. The idea is to choose a complete set of polynomials in which to expand the solution. We choose the Chebyshev polynomials $T_i(x)$. Consider a equation

$$L(\phi(x)) = \frac{\partial^2 \phi(x)}{\partial x \partial x} + x^2 \phi(x) = 0$$

Now we expand the solution in Chebyshev polynomials and truncate the basis to ‘n’ first polynomials only

$$\phi(x) = \sum_{i=0}^{n} a_i T_i(x)$$

where $\{a_i\}$ are coefficients we will solve for. This truncated basis spans a proper subspace in the space of smooth functions, which makes this method an approximation scheme. To proceed, we plug this ansatz to the original equation and obtain

$$\sum_{i=0}^{n} a_i L(T_i(x)) = 0$$

The left hand side can be expanded in Chebyshev polynomials and then the equality implies that the coefficient of each Chebyshev polynomial should vanish separately due to the orthogonality of this basis. This can be expressed as a matrix equation

$$\mathcal{M} \cdot a = 0$$

where

$$\mathcal{M} = (T_j(x),\ L(T_i(x)))$$
$$a = \{a_0,\ a_1,\ a_2,\ \ldots,\ a_n\}.$$

The brackets denote the orthonormal inner product on the space of basis functions. Then to solve for ‘a’ we can simply invert $\mathcal{M}$.

a) Implement the inner product of Chebyshev polynomials. Make sure that the Chebyshev polynomials are orthonormal with respect to your inner product (hint: see wikipedia).

b) Check that you can compute the inner product of any function with a Chebyshev polynomial. That is $(T_i(x), f(x))$. Assemble the matrix $\mathcal{M}$.

c) Is your matrix invertible? Why/why not?

d) Now impose initial conditions for ϕ[-1]==1, ϕ[1]==0 (you can choose other conditions also). Do this by evaluating the ansatz at the given value of x and read what are the coefficients of a_i. Insert the two equations you obtained to the last two rows of the matrix equation.

Invert the matrix equation and obtain the coefficients $\{a_i\}$. Plot the solution. Compare this scheme with NDSolve.