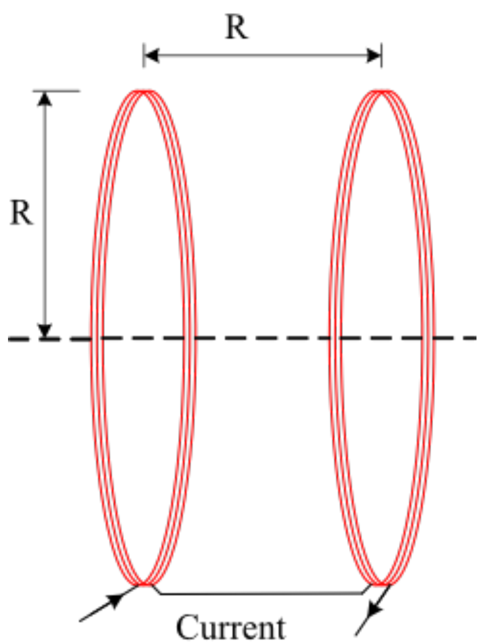
Electrodynamics

1. Helmholtz coil

The following configuration of current carrying wire is called a Helmholtz coil.



Helmholtz coil is useful because it produces a particularly uniform magnetic field on its z-axis (dashed line). The magnetic field on the z-axis is

$$B_z(z) = \frac{N \mu_o I R^2}{2}\left(\frac{1}{\left((z-R/2)^2+R^2\right)^{3/2}} + \frac{1}{\left((z+R/2)^2+R^2\right)^{3/2}}\right)$$

where N is the number of turns in the wire loops and $z = 0$ is located in the middle of the system. Show that the first and second derivatives of $B_z(z)$ at $z = 0$ are zero.

a) Show that the first and second derivatives of $B_z(z)$ at $z = 0$ are zero.

b) Show that the Taylor series of $B_z(z)$ at $z = 0$ is

c) Write a function that produces a plot of $B_z(z)$ and the nth order Taylor series of $B_z(z)$. How well does the series approximation fit the actual magnetic field?

d) Use Manipulate to produce an interactive version of the previous plot you made where you can change the order of the series approximation interactively

2. Magnetic dipole

a) Show that the magnetic field of a dipole is

$$\overline{B}(\overline{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\overline{m}\cdot\overline{r})\overline{r}}{r^5} - \frac{\overline{m}}{r^3}\right),$$

starting from the vector potential

$$\overline{A}(\overline{r}) = \frac{\mu_0}{4\pi}\frac{\overline{m}\times\overline{r}}{r^3}$$

Hint: calculate the curl of $\overline{A}$

b) Produce a plot of the dipole magnetic field. Hint: look up StreamPlot or VectorPlot

3. Plane wave polarizations

Maxwell equations admit plane wave solutions of the form

$$\overline{E}(\overline{r}, t) = E_x \cos(\omega t - \overline{k}\cdot\overline{r} - \phi)\hat{x} + E_y \cos(\omega t - \overline{k}\cdot\overline{r})\hat{y},$$

where ω is the angular velocity, $\overline{k}$ is the wave vector and ϕ is the phase difference between components. This solution represents a plane wave propagating in the z-direction. In general the electric field draws an ellipse on the (x, y)-plane. We'll analyze different polarizations of the wave at $\overline{r} = 0$.

a) First set $E_x = E_y$ and phase difference $\phi = \frac{\pi}{2}$. This corresponds to spherical polarization. Make a plot of the shape drawn on the (x, y)-plane by $\overline{E}$. Hint: ParametricPlot

b) What happens if you change the phase difference ϕ to some other value?

c) Use Manipulate to create an interactive panel where you can see a plot like the ones in previous parts but you also can change values of E_x, E_y and ϕ.

4. Lorentz transformations

a) Define matrix-valued function that gives a Lorentz boost for a given velocity.

You can either specialize to boosts in the x-direction or work with a general boost matrix, your choice. You may work in units of c if you want.

b) Check that your Lorentz boosts form a group. That is, check that two successive Lorentz boosts with velocities v_1 and v_2 is a Lorentz boost. Also you have to check that for each boost, there is an another boost which can undo the first boost.

c) Then, google the electromagnetic field tensor and define a matrix corresponding to it. Show that

$$F_{\mu\nu} F^{\mu\nu} = 2\,\overline{E}\cdot\overline{E}/c^2 - 2\,\overline{B}\cdot\overline{B}$$

using direct matrix operations. Remember to take into account the contra/covariant indices.