Mechanics

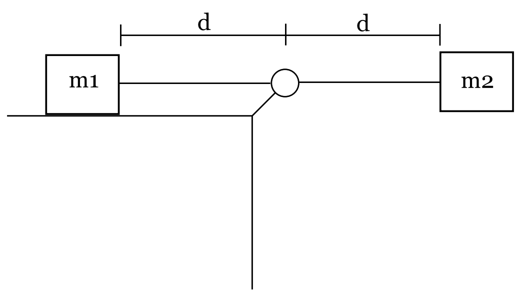
Note: these problems can be solve mostly or completely by hand but try to do all the manipulations using *Mathematica*. If you are unsure of your answer, you can check using pen and paper.

The Magnus effect

Go watch https://www.youtube.com/watch?v=QtP_bh2IMXc. Analyze the situation of a falling spinning object with a drag. Include the magnus force, F=D $\omega \times v$ and two kinds of drags, F~ -v and F~ -v^2. What is the terminal velocity? (analytically). What is the initial behaviour of x and y when they start from rest i.e. solve α and β in (x[t]-x[0])~t^α, (y[t]-y[0])~t^β (analytically).

Solve the equations numerically and plot the results and check whether your analytical predictions were correct.

Race to the finish



Consider the following problem where two block are attached by a string of length 2d. The two blocks are initially at rest. The object 2 starts falling and object 1 starts sliding frictionlessly towards the pulley. Set m1=m2. Which block finishes first? (I.e. m2 hits the wall or m1 hits the pulley). Or is it a tie?

Check the results using numerics. Can you change the results qualitatively by adjusting the masses, d or g?

Feel free to use either Lagrangian or Newtonian formalism.

Spherical pendulum

Consider a point particle (mass ‘m’) confined on a sphere of radius ‘l’ in a gravitational field. If we choose spherical coordinates, the Lagrangian of this system is

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L := 1/2 m l^2 (θ'[t]^2 + ϕ'[t]^2 Sin[θ[t]]^2) - m g l Cos[θ[t]]
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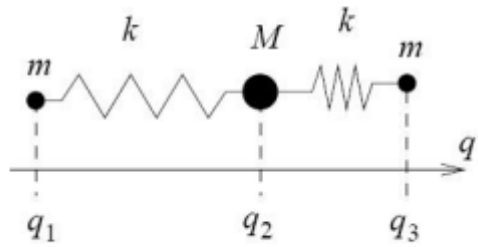
a) Derive the equations of motion from the Lagrangian.

b) Set g→1 and m→1 (bonus: explain why you can do this). Solve the equations of motion numerically for a given ‘l’ and plot the result using ParametricPlot3D.

c) Use Manipulate to illustrate how the particle moves in time.

Oscillating molecule

Now consider three point particles of mass ‘m’, ‘M’ and ‘m’ connected together by springs of length ‘b’.



a) Write down the Lagrangian of this system.

b) Define $x_i[t] := q_i[t] - q_i^{(0)}$, where $q_i^{(0)}$ is the equilibrium coordinate of the i_{th} particle. Note that $b = q_2^{(0)} - q_1^{(0)} = q_3^{(0)} - q_2^{(0)}$. Manipulate the Lagrangian to the following form

and assemble the matrices $\mathcal{M} := (m_{ij})$ and $\mathcal{K} := (k_{ij})$. Note the factors of 1/2 in the above Lagrangian.

At this point you should have $\mathcal{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$ and

$\mathcal{K} = k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

c) The ansatz $x_i = C\, a_j\, e^{i\,\omega\,t}$ (C is constant) brings the equations of motion to form $(\mathcal{K} - \omega^2\, \mathcal{M}) \cdot a = 0$, where $a = \{a_1, a_2, a_3\}$. Solve for eigenvalues ${\omega_i}^2$ and eigenvectors a_i. Normalize the vectors such that ${a_i}^{\mathsf{T}} \cdot \mathcal{M} \cdot a_i = 1$.

d) Now define $\mathcal{A} := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Normal modes are now x=$\mathcal{A} \cdot \eta$, where $\eta = \{\eta_1, \eta_2, \eta_3\}$ are the normal coordinates. Visualize how these normal modes oscillate in time. That is, first set $\eta_2 = \eta_3 = 0$ and see what the oscillation looks like when only the first normal coordinate is non-zero, etc.

bonus) Add more point masses and see what the normal modes look like.

For details about the physics of this problem, see the book ‘Klassinen mekaniikka’ by Koskinen and Vainio (or any other classical mechanics book).

Poisson brackets

The Poisson bracket is a binary operation on the phase space of a classical system with the following properties
{f,g}=-{g,f}
{f+g,h}={f,h}+{g,h}
{f g,h}={f,h}g+f{g,h}
{c,h}=0 (if c is constant)

a) Create a prettier notation for your Poisson bracket using the Notation-package.

b) Implement a Poisson bracket with these properties. Do not use a coordinate representation for {f,g}.

c) It is a very similar exercise to implement the algebra of a 1D quantum harmonic oscillator. Go back to problem set 1 and do the 1D oscillator problem if you haven’t already.