Final exercise in differential geometry and general relativity.

Ex I:

Show that the metric $ds^2 = \frac{L^2\, dr\text{^}2}{r^2} + \frac{r^2\left(-dt^2+dx^2+dy^2+dz^2\right)}{L^2}$ is a solution to the Einstein equations with a cosmological constant, $\Lambda=-\frac{6}{L^2}$, $\text{Einstein}_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0$. Then, add the electromagnetic field tensor with one non-zero component: A={0,At[r],0,0,0} and solve the Einstein equations: $\text{Einstein}_{\mu\nu} + \Lambda\, g_{\mu\nu} = T_{\mu\nu}$. Use the ansatz metric = DiagonalMatrix[{L^2/r^2 g[r],r^2/L^2 f[r],r^2/L^2,r^2/L^2,r^2/L^2}]. Find a charged black hole solution.

Ex II:

(Following an exercise by L. Yaffe). Our main goal is to find quasinormal modes of the AdS_5 spacetime with a black hole. First, we consider the Eddington-Finkelstein coordinates. Use the metric: $ds\text{^}2 = 2\, dt\, (dr - A[r]\, dt) + \frac{r^2\left(dx^2+dy^2+dz^2\right)}{L^2}$. Find the most general A, such that it solves the Einstein equations with a cosmological constant $\text{Einstein}_{\mu\nu} + \Lambda\, g_{\mu\nu} = 0$. Here Λ= -6/L^2.

You should obtain: $A[r] = \frac{r^2}{2\, L^2} + \frac{C}{r^2}$. Replace the constant C with $-\frac{m\, L^2}{2}$. Now, solve the horizon radius of this solution, i.e. find r_H s.t. $A[r_H]=0$. Also, solve the Hawking temperature of this metric, i.e. $A'[r_H]/(2\pi)$. Then, solve m in terms of the Hawking temperature and create a replacement rule “msub” (or something).

Ex III:

To obtain quasinormal modes, we need to add plane wave -like features to our metric. To the metric used and solved in Ex. II, add a non-zero component: $g_{xy}=\eta\, e^{i\,(q\, z-\omega\, t)}\, \frac{r^2}{L^2}\, f[r]$, where η is small. Recompute Einstein tensor and such.

Compute the Einstein equations: Einstein + Λ g to linear order in η. Obtain the equation of motion for f[r]. We will refer to it as QNMeqn.

You should get:
$L^2\, r\, \left(L^2\, q^2 + 3\, i\, r\, \omega\right) f[r] + \left(-5\, r^4 + 2\, i\, L^2\, r^3\, \omega - 2\, L^2\, C[1]\right) f'[r] - r\, \left(r^4 + 2\, L^2\, C[1]\right) f''[r] ==$
0.

Try solving the equation. See that the possible result is useless.

Show that QNMeqn has singular points at $r = r_H$ and r=∞.

Now, solve the behaviour of f[r] near r_H and r=∞. That is, make the ansätze $f[r] = (r - r_H)$^γ(1+O[r-rH]) and f[r]=r^(α)*(1+O[1/r]), respectively, and solve for γ and α. You should obtain two solutions for both α and γ

Ex IV:

Now for the hard part. It turns out that the solution for f[r] exists only for some discrete ω with any given q^2. We are after these discrete values of ω, the quasinormal modes. We will employ spectral methods, i.e. we expand our function-to-be-solved as a sum of Chebyshev polynomials of the first kind (ChebyshevT) which we know to be orthogonal from previous exercises.

First, we set $f[r] = g[r / r_H]$ r^(-4). Now g is defined between 0 and 1 and should be regular. Why?

Plug this ansatz for f[r] into QNMeqn and massage it into a non-vanishing and non-singular form. Do the substitutions q = k (2 π T_H) and ω = λ (2 π T_H) and use “msub” to get the following equation for g:
$\left(4\, k^2\, u + 16\, u^3 - 10\, i\, \lambda\right) g[u] + \left(-5 + 9\, u^4 - 4\, i\, u\, \lambda\right) g'[u] + u\, \left(-1 + u^4\right) g''[u] == 0.$

Choose some integer value for M, between 4 and 8. Chebyshev polynomials take x values between -1 and 1 and at grid points given by Cos[l π /M], l=0,1,..., M. Write replacement rule for g, expanding it as a linear combination of Chebyshev polynomials upto the ChebyshevT[M,x] (in total, M+1 terms). Use pure functions so that the derivatives are transformed correctly. Remember to scale the arguments properly.

Also, write a replacement rule for u such that it replaces u with the grid point. It could be something like: {u→Cos[...]}. Remember to scale and shift if necessary. Also, evaluate the grid points numerically with “N” to help with later computations.

Replace both g and u in your equation of motion for g to obtain M+1 equations for the coefficients in the expansion of g.

Transform this system of equations into a matrix. We call it “Q”.

Elements of Q depend on λ at most linearly. Split Q to α and β: Q= α - λ β, define α and β.

Solve the generalized eigenvalue equation for first few λ, α.v=λ β.v. Pick some numeric value for k, I suggest k=1. Eigenvalues is the correct function to do this, check the documentation. Why is the standard way computing the determinant of “Q” and solving the equation for λ a bad choice here? (I.e. why is it slow?)

Combine the work of Ex. IV into a few cells to compute eigenvalues for arbitrary M and k. Increase M and see how the first few (six) eigenvalues converge.

Bonus: You should see that there appears to be some numerical inaccuracy in computing the eigenvalues. There are a few ways to fix this:

1. Increase numerical precision when evaluating the grid points, with N[expr,precision]. Try precision upto 30 or maybe even 50.

2. The main source of inaccuracy is in evaluating the Chebyshev polynomials and it’s derivatives. The source of the problem is that when we first introduce the Chebyshev polynomials with some integer first argument but symbolic second argument, *Mathematica* evaluates the expression to a polynomial expression. Later, when we substitute the grid points, large cancellations occur which lead to the accuracy. We fix this by expanding g in our own set of functions, say ChebyshevTT[n_,x_]. Later on, when defining the matrices, we replace ChebyshevTT[n_,x_]-> ChebyshevT[n,x]. This needs to be done separately for the first and second derivative. We can use Gegenbauer polynomials (GegenbauerC) for which the following is true: D[ChebyshevT[n,x],{x,m}] == n 2^(m-1) (m-1)! GegenbauerC[n-m,m,x]. Check that this is true. Definitions may vary.

Bonus: Increase M upto 40 and numerical precison upto 40 and compare the ten first eigenvalues to those of Kovtun and Starinets in arXiv:hep-th/0506184. See Appendix B, last table, left column.