Project: Classical mechanics with an RK4 integrator

As we have seen, many differential equations have to be solved numerically. This project has two parts. The first part consists of implementing your own numerical integrator using the Runge-Kutta method. The second part involves using your implementation to study two kinds of classic problems of classical mechanics. The first is about deriving the equations of motion for two coupled pendulums and studying its chaotic properties. The second is to study the gravitational 3-body problem.

Runge-Kutta (RK4) integrator

The Runge-Kutta integrator solves a system of 1st order differential equations, $\partial_t g[t]$=A[g,t] with an initial condition g[t0]=.... The solution is provided step-by-step, i.e. g[t0], g[t0+δt], g[t0+2 δt],.... To solve 2nd order differential equations, we need to define a new function to solve. For example, $\partial_t \partial_t$f+∂_tf+f=0 can be written equivalently as $\partial_t \phi$ = -f-ϕ and ∂_tf=ϕ. Now we have two 1st order differential equations. The initial conditions also need modifications.

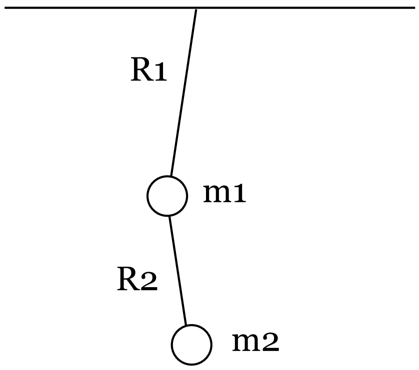
1. Construct a step-function using the RK4 method (c.f. Wikipedia). It should take in the current state of the system, current time, size of the step and the equation defining the system. It returns δg in g[t+δt]=g[t]+δg.

2. Construct an integrator using your step-function. It should take in the initial condition, initial time, the equation of motion, size of the step and either the finishing time or the amount of time steps to take. It returns a list $\{\{t0, g[t0]\}, \{t0+\delta t, g[t0]+\delta g_1\}, \{t0+2\delta t, g[t0]+\delta g_1+\delta g_2\}\ldots\}$

3. Solve the differential equation y’’=-y with initial condition y(0)=0, y’(0)=1 and compare your RK4 integrator results with the analytical solution. You should get very good agreement.

4. Try it on y''=-x y as well with the same initial conditions. The agreement should be significantly worse but still relatively good.

Applications to two coupled pendulums



Consider a system of two coupled pendulums. One is fixed to a point in the ceiling and the other pendulum is hanging from the first pendulum. The strings between the objects have a constant length and they are always rigid. The systems is chaotic in nature. Some choices of coordinates might be better than others.

5. Write down the Lagrangian of the system with appropriate constaints and Lagrange multipliers.

6. Derive the Euler-Lagrange equations of motion and manipulate them until you have two remaining equations of motion (both 2nd order).

7. Prepare your equations of motion for your RK4 integrator.

8. Solve one simple non-trivial case with some arbitrary values. Animate your results.

8. Study chaos within this system: Track the motion of the smaller mass (e.g. with ListPlot or Animate) with two slightly different initial conditions (e.g. shift the starting position or initial velocity of mass two a tiny bit to the right). Do at least two different comparisons. Things to consider: make masses vastly different, adjust the length of the strings etc. When does the motion diverge significantly (describe the phenomenon)

Applications to gravitational 3-body problem.

The three-body (and n-body) gravitational problem was the topic of the mathematics competition to celebrate the 60th birthday King Oscar of Sweden and Norway in the late 19th century. Nobody was quite able to solve it before the king’s death in 1905. Later, a series expansion solution to the 3-body problem was found by the Finnish K. F. Sundman in 1906 and 1909.
However, before this solution, Poincaré had discovered the chaotic nature of the system.
We will now study this problem with the tool that we developed.

9. Consider 3 gravitational bodies in a plane. Write down the equations of motion using whatever method you found suitable.

10. Solve numerically and animate the following cases:

a) One star and two planets with stable orbits

b) One star, one planet with a moon

c) Three bodies with crossing trajectories