Problem set 2

Easy Problems

Solving equations

1. Solve the equation $x^2 + 3x - 10 == 0$ for x and verify the roots by plotting the left hand side of the equation.

2. Solve the following system of equations $\{x\,y == 1/2,\ x^2 + y^2 == 1\}$ for (x,y).

3. Find the second order polynomial $p(x) := a\,x^2 + b\,x + c$ that satisfies $\{p(-1) == 2, p(2) == -1, p(5) == 0\}$ and plot the resulting polynomial.

4. Solve the equation $\text{Cos}[x] == x$ for x. *Mathematica* refuses to yield an answer, why?

5. Use FindRoot to find the first positive root of the previous equation.

Integration and differentation

6. Define any simple function of x. Compute its indefinite integral w.r.t. x. Compute the derivative to verify you get back your initial function.

7. Integrate the function $x\,e^{-x}$ from 0 to x'. Plot the value of the integral w.r.t. x' and find x' s.t. the integral yields $\frac{2}{3}$.

8. Compute the integral of $\frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}$ (ChebyshevT) from -1 to 1 for first few n and m. What property is implied for the Chebyshev-functions?

9. Find the points where the derivative of $1 - 2x + \frac{1}{2}x^3$ changes sign.

10. Demonstrate with $f(x, y) = \frac{x\,y\,(x^2-y^2)}{x^2+y^2}$ that partial derivatives commute, that is, $\partial_x \partial_y f == \partial_y \partial_x f$.

11. Compute the surface integral $\int x^2\,y\,\text{dxdy}$ over a region defined by $0 \le x \le 1\big/\sqrt{2},\ x \le y \le \sqrt{1-x^2}$.

Series expansion

12. Compute the series expansion of e^x around x=0 and confirm the result is

what you expected.

13. Compute the series expansion of Sin[x] around x=0 to all orders from 1 to 10 (separately). Plot the series approximations and the exact function in a same figure.

14. Find the coefficient of the x^{-1}-term in the series expansion of $\frac{\text{Sin}[x]}{x(x-1)}$ first at x=0 and then x=1.

15. Find the asymptotic behavior of $K_n(x)$ (BesselK) at infinity x→∞.

16. Expand the expression $\sqrt{1+y^2+x\,y}$ to first order in both x and y. Here both x and y are considered 1st order quantities, so for example, x y is second order and should be discarded (hint: replace x→ϵ x, y→ϵ y).

Differential equations

17. Solve the linear differential equation $y'[x]+3\,y[x]==x$ and plot a family of solutions with different values of the integration constant.

18. Solve the nonlinear differential equation $y'[x]==\frac{y[x]}{x}+\frac{x}{2\,y[x]}$, $y[2]==0$ and plot your solution.

19. Solve the system of equations $x'[t]==\text{Cos}[z[t]]$, $y'[t]==\text{Sin}[z[t]]$, $z'[t]==t$ with initial conditions $x[0]==0$, $y[0]==0$, $z[0]==0$ for the functions $x[t]$, $y[t]$ and $z[t]$. Use ParametricPlot to plot {x[t],y[t]} w.r.t. t.

20. Use NDSolve to solve the one-dimensional wave equation $-\partial_{t,t}\phi[t,x]+\partial_{x,x}\phi[t,x]==0$ in a box x={-1,1} with Dirichlet boundary conditions ϕ[t,-1]==0, ϕ[t,1]==0 and initial condition ϕ[0,x]=$\cos\left[\frac{x\pi}{2}\right]e^{-(5x)^2}$. Use Plot3D to visualize your solution.

Recursion and sequence problems

21. Use RSolve to solve recursive equation: {a[n+1]==-1/2 a[n]+n, a[1]==1}.

22. Use RSolve to solve recursive equation: {a[n+2]== a[n+1]+a[n], a[1]==1,a[2]==1}.

23. Find the explicit form for Fibonacci numbers. (Hint: solve a generalized version of the defining recursive equation)

24. Implement function MyFactorial, that reproduces the behaviour of Factorial for integer entries. I.e. some intended outputs would be MyFactorial[0]=1, MyFactorial[5]=120, MyFactorial[1/2]=MyFactorial[1/2],

MyFactorial[x]=MyFactorial[x] (x undefined).

25. Implement function MyFibonacci, that reproduces the behaviour of Fibonacci. Use Timing to measure the speed of your implementation. Try n=10,20,30,40,50 (Hint: Alt+. to abort evaluation)

26. In case of slow evaluation, implement MyFibonacci using memoization, i.e. the previously evaluated function values are stored in memory. MyFibonacci[n_]:=MyFibonacci[n]=original definition. Use timing on n=50,150,300,1000. Type ?MyFibonacci. Do you see anything negative with your implementation? (See harder problems)

Medium and harder problems

27. Generate a list of 10 equations of the form $a_1 x_1 + a_2 x_2 + \ldots a_{10} x_{10} == b$ for random a_n and b. Represent this system of equations as a single matrix equation and solve it for a_n. Check with Solve that your result is correct.

28. Consider the following system of equations (copy&paste to your notebook)

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u == a + (x - a) (1 + c ((x - a) (x - a) + (y - b) (y - b)))
v == b + (y - b) (1 + d ((x - a) (x - a) + (y - b) (y - b)))
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and visualize the roots (x,y) with ContourPlot. Set the constants (a,b,c,d) to any values between (-1,1). Use Manipulate to see how the solutions behave when the values (a,b,c,d) are changed.

30. Integrate 1 (unity) over the unit circle. You should obtain π.

31. Implement your own function for differentiation including linearity, chain and product rules. Add also the derivatives of Sin, Cos, Tan, Exp and Log.

32. Create a function ‘Children’ which returns the list of children of its argument without evaluation. For instance, Children[h[a,b]]→{a,b} and Children[2+2]→{2,2}.

33. Sometimes one has to deal with sums that converge awfully slow. One famous example is the sum $\sum_{k=0}^{\infty} \frac{4\,(-1)^k}{2\,k+1}=\pi$. One can accelerate the convergence of the sum using various methods. One simple example is the Shanks transformation (originally by Schmidt). Define the partial sum: $A_n=\sum_{k=0}^{n} \frac{4\,(-1)^k}{2\,k+1}$, which itself is also a sequence. Now, the Shanks transformation is $S(A_n) = \frac{A_{n+1} A_{n-1} - A_n^2}{A_{n-1} + A_{n+1} - 2 A_n}$, which is also a sequence. Implement the

Shanks transformation to at least eight levels (i.e. $S(S(S(S(S(S(S(S(A_n))))))))$) and compute it. Plot the convergence of the error when comparing to the known result. Try it also with other slowly converging sums. What happens with diverging sums?

34. Find the nth derivative of x[t_]:= A Sin[ω t+ ϕ] (c.f. question asked during lecture 2). (Hint: compute the first 10 derivatives of the function and use FindSequenceFunction to find the general form.)

35. Do a fast implementation of Fibonacci (MyFibonacci) without using memoization. Use an auxiliary function.

36. We try to find a perturbative solution to equation: x^5+x-1==0. We set a perturbation parameter ϵ at x^5+ϵ x-1==0. When ϵ==0, it has the trivial solution x→1. We try to find a solution to the perturbative equation using the expansion, x= $1+\sum_{i=1}^{\infty} \epsilon^i$ a[i]. The solution to the full equation should then be recovered by setting ϵ=1. Expand the equation around ϵ=0 using this ansatz and solve the coefficients upto a[20]. Estimate the radius of convergence.

37. Repeat analysis for ϵ x^5+x-1==0. What is the radius of convergence? Why does this method not work?