Problem set 3

Easy Problems

Notation and vectors

1. Create a notation $\underset{\sim}{v}$ to represent `Vector[v]`.

2. Compute the matrix products, XY and YX, where X = $\begin{pmatrix} 2 & 3 & 7 \\ 0 & 9 & 3 \\ 5 & 0 & 0 \\ 1 & 0 & 2 \end{pmatrix}$, Y=$\begin{pmatrix} 2 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$.

Show explicitly that Tr[XY]=Tr[YX].

3. Solve the matrix equation for v: A.v=B.b, where A = $\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 2 \end{pmatrix}$, B = $\begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, b = $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

4. Create a random 10x10 integer matrix with entries between -10 and 10. Compute the eigenvalues to MachinePrecision.

5. Construct a diagonal matrix with the diagnonal elements Cos[l π / N], l =0,1,...,N. Do this in two different ways.

6. Construct the following matrix: $\begin{pmatrix} J & 0 & A \\ 0 & J & 0 \\ A & 0 & J \end{pmatrix}$, where each element is a 3x3 matrix, J = $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, A = $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Use block matrix format and ArrayFlatten.

7. Construct any non-trivial nxn matrix. Decompose it into (mxm), (n-m x n-m), (1n - m x m) and (m x n-m) matrices, so that you could reconstruct the matrix with the block matrix format.

8. Suppose T = $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. Compute the matrix exponential, i.e. $e^T = 1 + T + T.T/2 + T.T.T/6 + \ldots$. Hint: Exp does not work here. What does it do?.

9. Solve the differential equation: ∂_tv[t]==$\begin{pmatrix} 1 & 1 & 2t \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.v[t], v[0]=={a1,a2,a3}. Show that e^(M t).v[0], where M=$\begin{pmatrix} 1 & 1 & t*(1-\frac{1}{t}+\text{Coth}[t]) \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is the solution.

Pure functions

10. Write the Gaussian density function, $\frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\,\sigma}$ as a pure function using two different methods.

11. Consider a generic 2nd order differential equation: f''+a(x)f'+b(x) f=0. Assume that we know one non-trivial solution to it, f1. Using f1, we can analytically compute a linearly independent solution: Using pure functions, substitute into the ODE f-> f1*g and solve g. Now, f1*g is another solution to the ODE.

12. Express the Laplace operator (in 3d) in cylindrical coordinates.

13. By default, (Full)Simplify minimizes the number of subexpressions and digits in integers. However, we can modify the complexity measure with (Full)Simplify[expression, ComplexityFunction→MyFunc]. Create a modified FullSimplify such that it minimizes the StringLength of the expression. Try it on Gamma[x]x and compare with FullSimplify.

14. Create a random list of pairs of numbers $\{x_i, y_i\}$ and sort them with the formula: x^3+π y.

Medium and harder problems

15. Create a noncommutative multiplication function that distributes over addition and factors out constants

16. Construct the following matrix: $H = \sum_{i=1}^{3}\sum_{k=1}^{N-1} \sigma_{i,k} \otimes \sigma_{i,k+1}$, where σ_i is the i^{th} Pauli matrix (PauliMatrix[i]) and ⊗ is a tensor product (KroneckerProduct). It is implied, that in each term there are N matrices to be (tensor) multiplied and the subindices of σ's indicate the position in the product, all the other matrices are 2x2 identity matrices. In other words, in the summand, there are k-1 and N-k-1 identity matrices before and after, respectively. The final matrix should be a 2^Nx2^N matrix. Plot the matrix for N=2,4,6,8 with MatrixPlot.

17. Consider a linear operator, Op, that maps basis vectors, $\{e_i, i \in \{1, 2, 3, \ldots\}\}$, in the following way: Op.e_k = -Sqrt[(k-1)k]* $e_{k-1} + k \wedge 2 * e_k -$ Sqrt$[k(k+1)] * e_{k+1}$. Construct the corresponding matrix, where i goes all the way until 50. (Hint: Use SparseArray and Normal.) This implies $e_{100+1} = 0$. Use MatrixPlot to visualize the matrix. Compute the ten lowest eigenvalues and eigenvectors (Hint: Eigensystem, also *Mathematica* is awfully slow with InfinitePrecision, try N). Plot the eigenvalues.

18. Construct a complex 2x2 matrix. Find the condition for it to be a special unitary matrix. I.e.write U = $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Demand ConjugateTranspose[U].U==IdentityMatrix[2] and Det[U]==1. Use Reduce to simplify the equations, this should give you simplified conditions.

19. Express the gradient operator (in 3d) in spherical coordinates