Using games for measuring similarity between mathematical structures

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DOMAST Student Seminar

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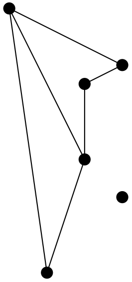
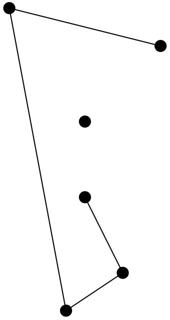
- for some fixed length n (a natural number) or ω (continues "forever")
- We denote by $\mathrm{EF}_n(\mathfrak{A}, \mathfrak{B})$ the EF game of length n between structures $\mathfrak{A}$ and $\mathfrak{B}$.

Example





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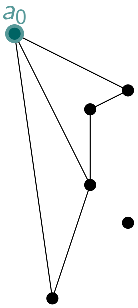

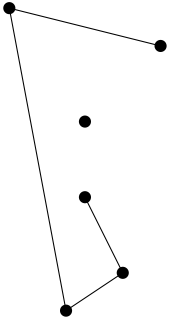

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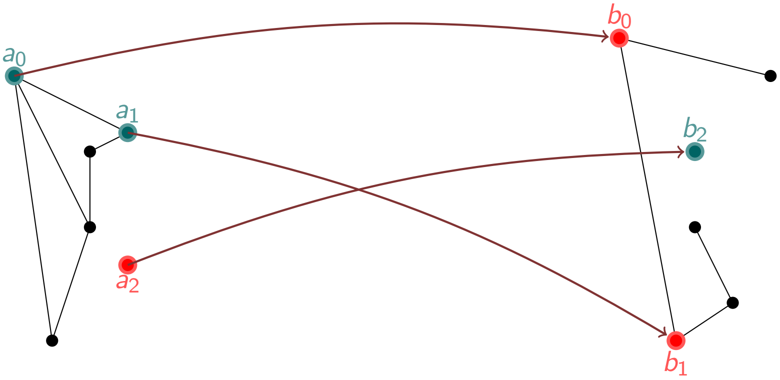

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  1. If $\mathfrak{A} = (G, \cdot)$ and $\mathfrak{B} = (G', \cdot)$ are groups, then $f\colon G \to G'$ is a partial isomorphism iff it is an injective partial homomorphism (in the algebraic sense)
  2. If $\mathfrak{A} = (A, <)$ and $\mathfrak{B} = (B, <)$ are linear orders, then $f\colon A \to B$ is a partial isomorphism iff for all $a, a' \in \mathrm{dom}(f)$

$$a < a' \iff f(a) < f(a').$$

Properties of the Game

- The game is determined: one of the players has a *winning strategy*
- If II wins a game of length n, then she wins the game of length m for any $m \leq n$
- If I wins a game of length n, then he wins the game of length m for any $m \geq n$

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Elementary Equivalence

Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* if they have the same *first-order theory*, i.e. for every (finitary) statement φ consisting of

- atomic statements, e.g. $x = y$ or $0 < n$,
- "and",
- "or",
- "not",
- "if ... then",
- "if and only if",
- "for every element x ...", and
- "there is an element x such that...",

φ is true in $\mathfrak{A}$ if and only if it is true in $\mathfrak{B}$.

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- “there are at least 7 elements”
- “there is a clique of 5 elements”, in the language of graphs
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"Non-elementary" properties:

- "there are only finitely many elements"
- "the graph is connected", in the language of graphs
- "every bounded non-empty set has a supremum", in the language of real closed fields

Theorem

$\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent if and only if, for every $n \in \mathbb{N}$, II has a winning strategy in the EF game of length n between $\mathfrak{A}$ and $\mathfrak{B}$.

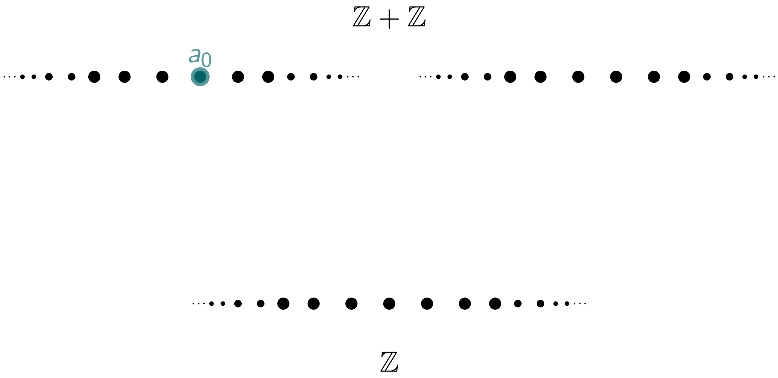
For proof, see e.g. J. Väänänen, *Models and Games*.

Example

$$\mathbb{Z} + \mathbb{Z}$$

$$\mathbb{Z}$$

Example



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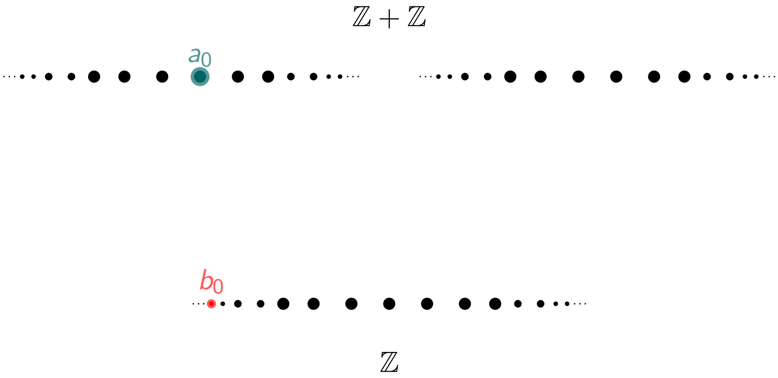

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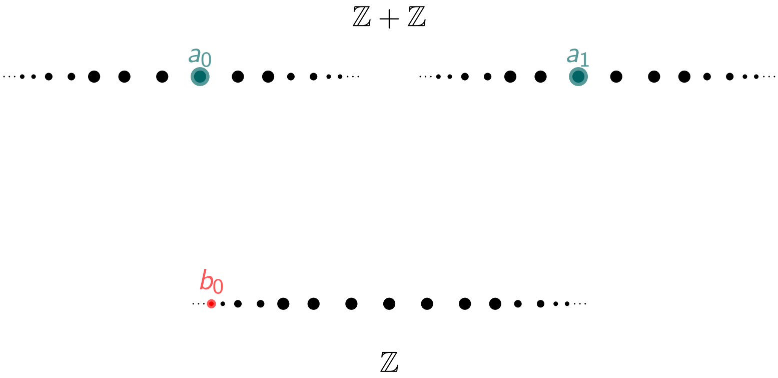

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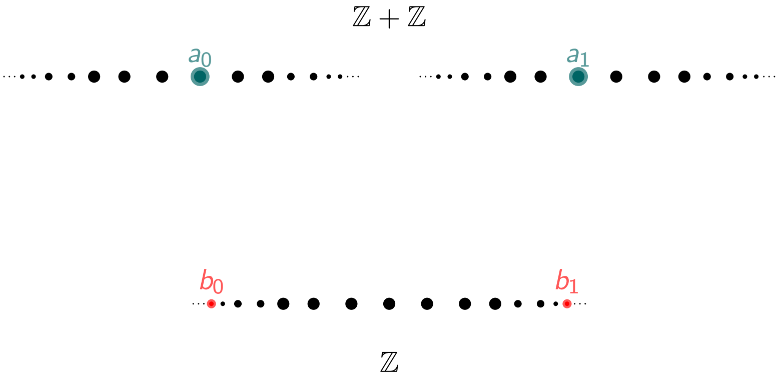

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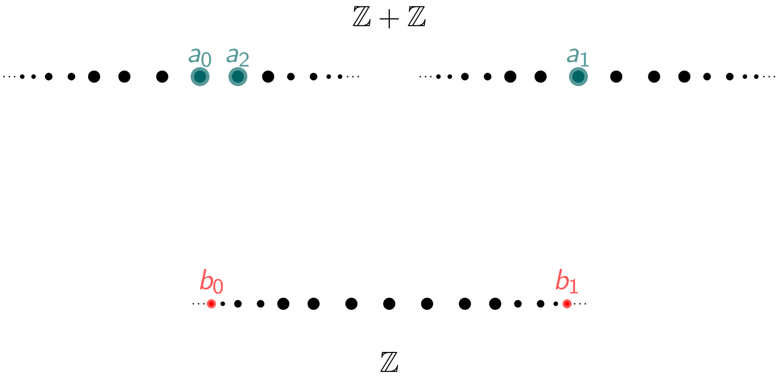

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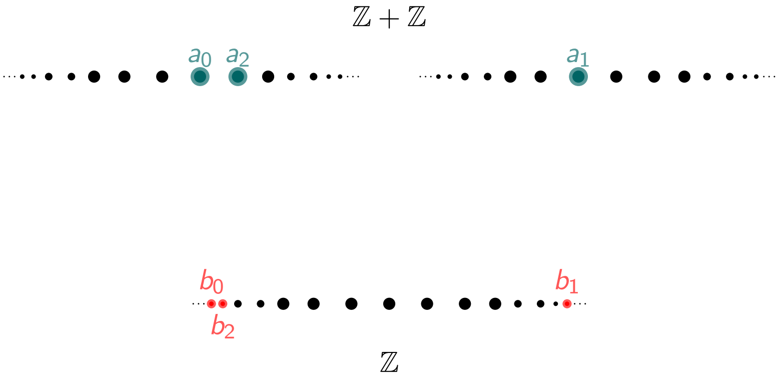

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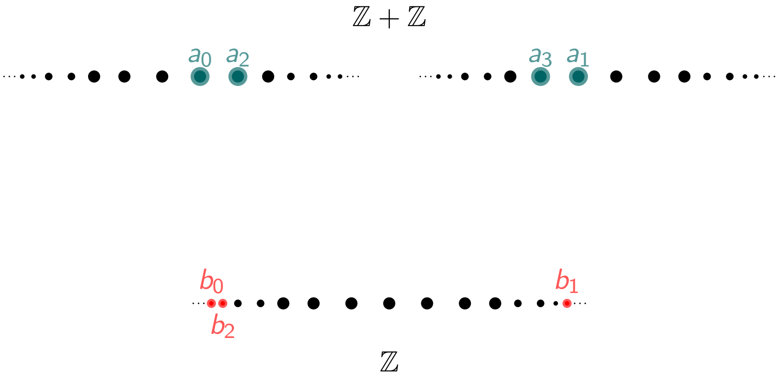

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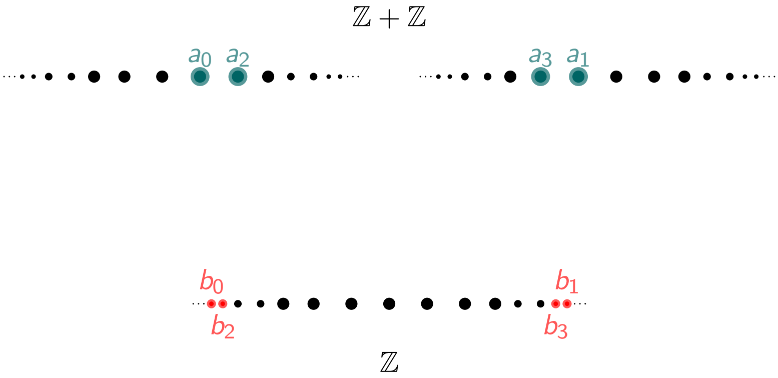

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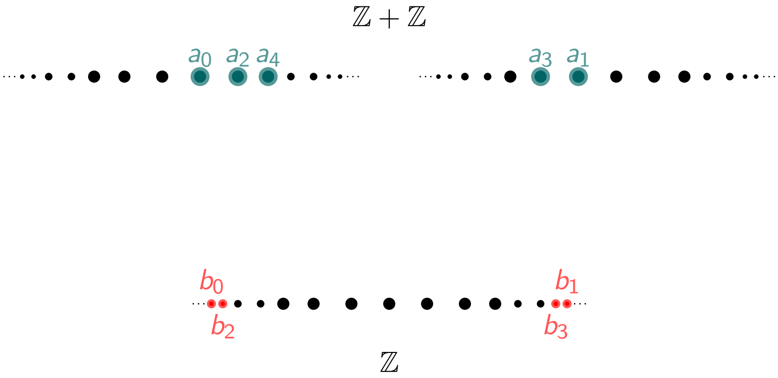

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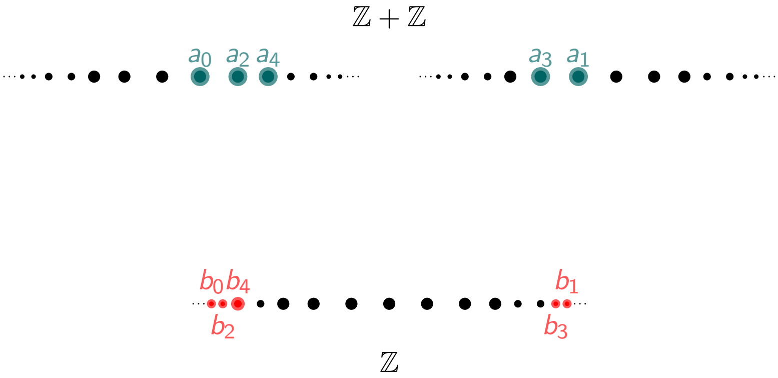

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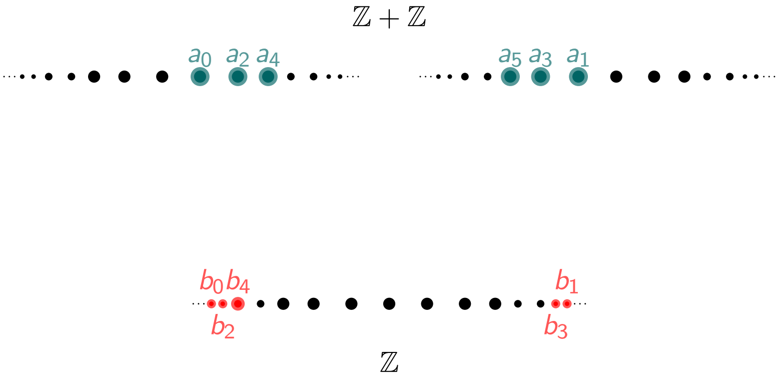

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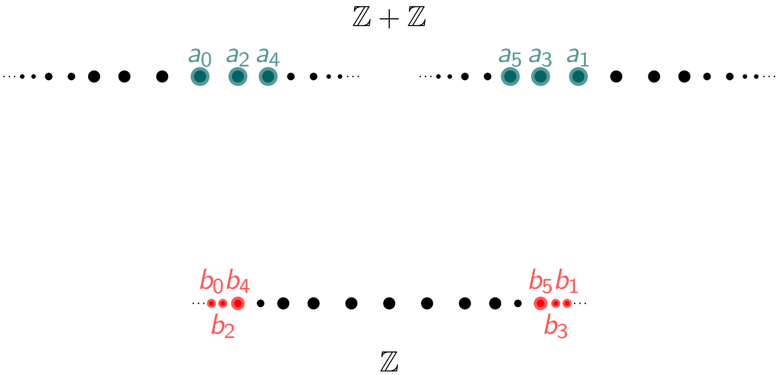

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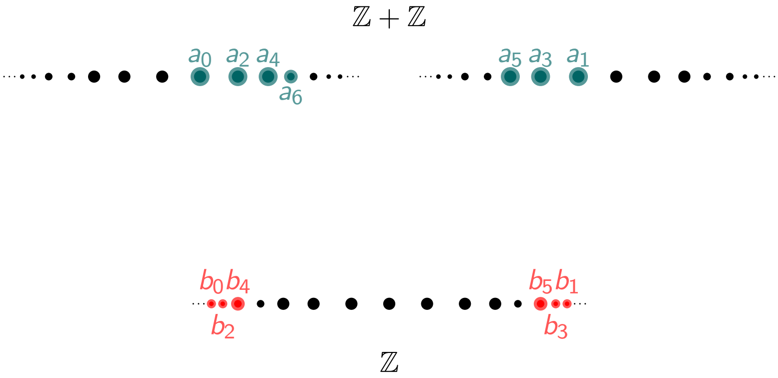

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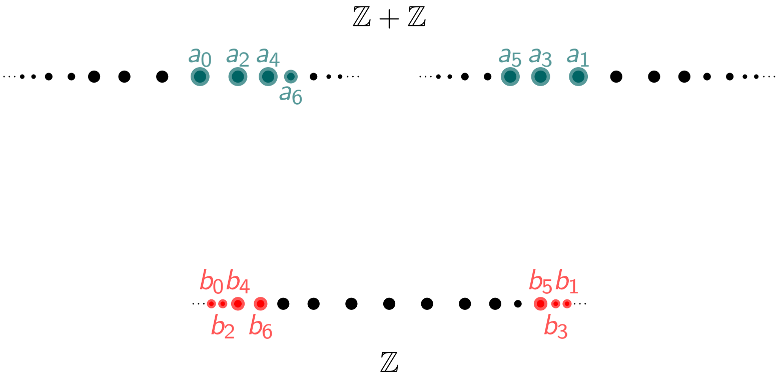

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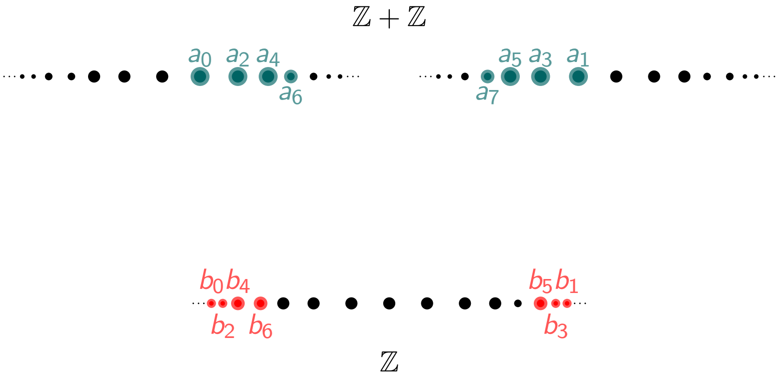

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a_0 a_2 a_4
a_6
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a_7
b_0 b_4
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b_5 b_1
b_3
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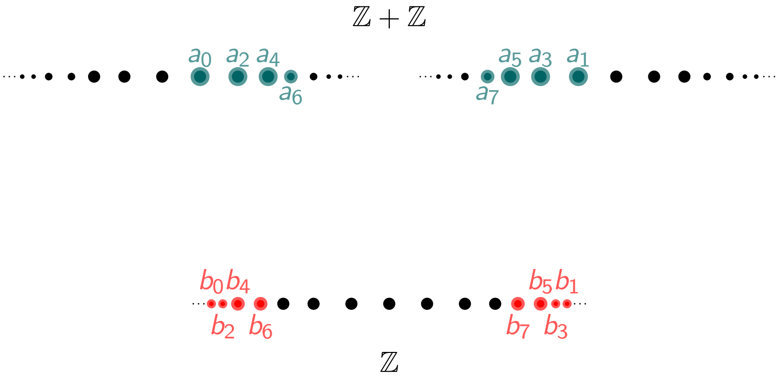
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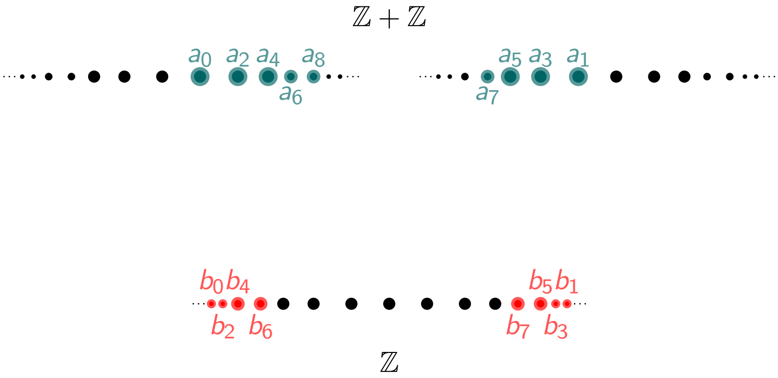
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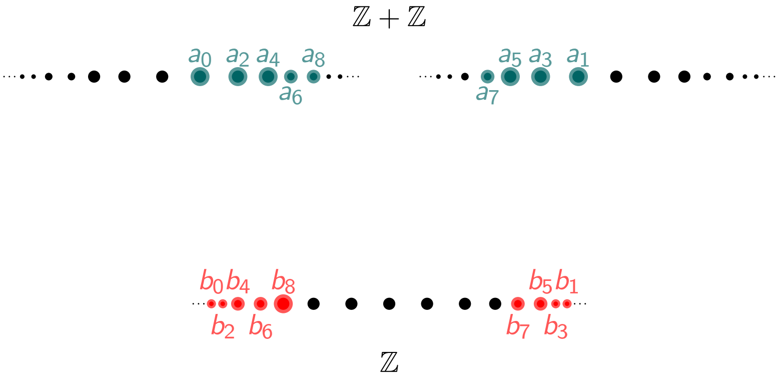

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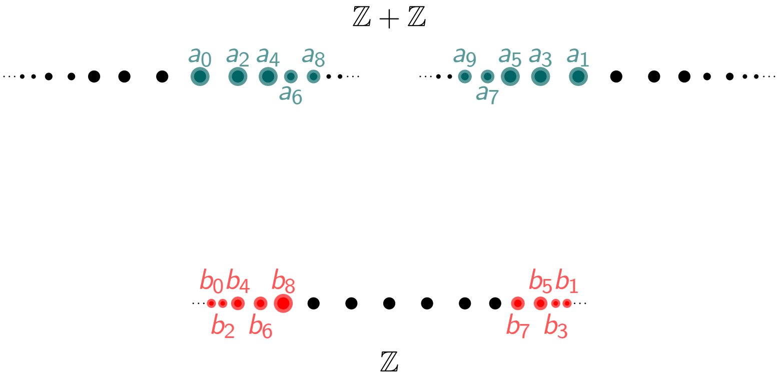

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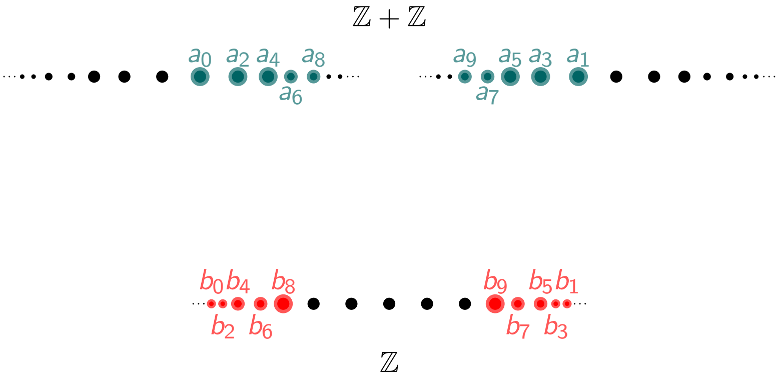
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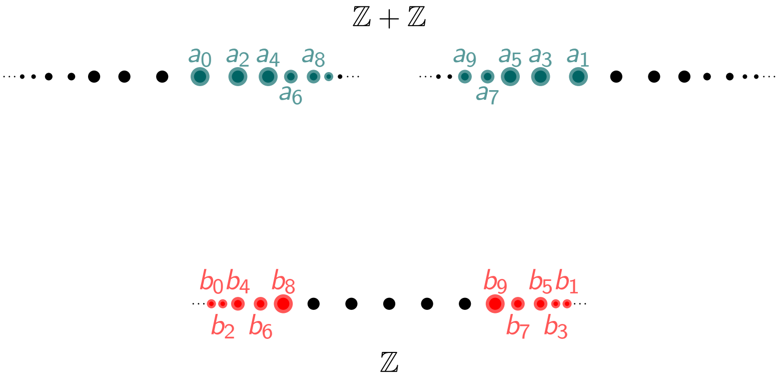
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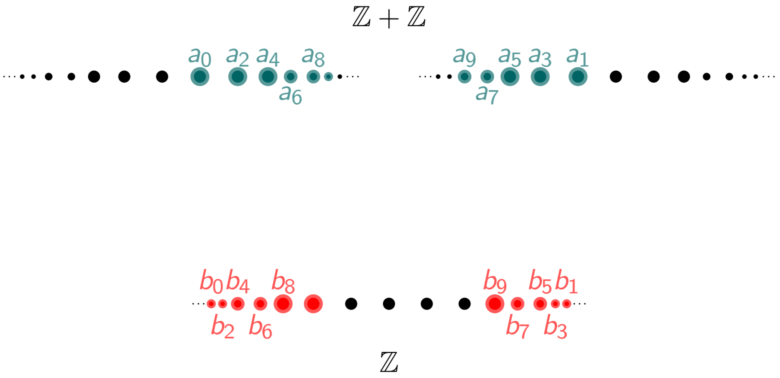
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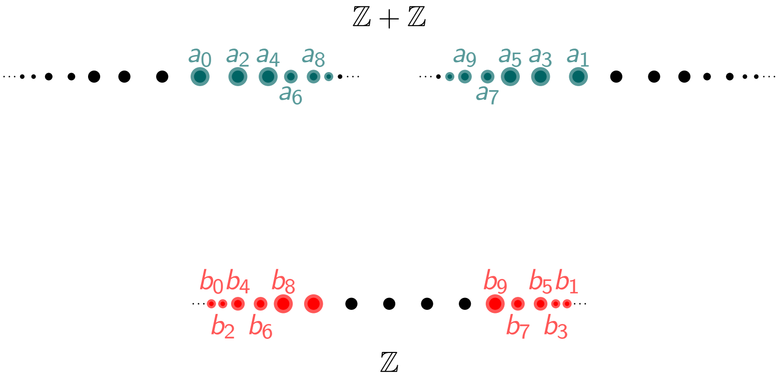

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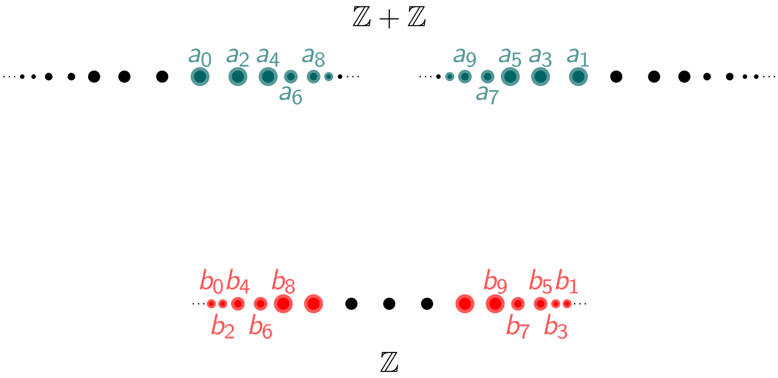

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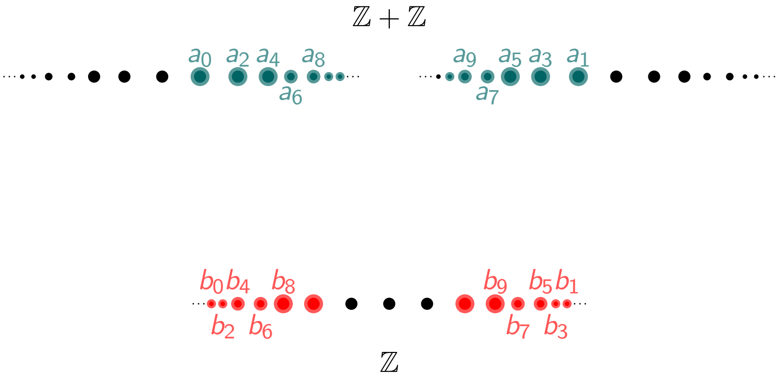

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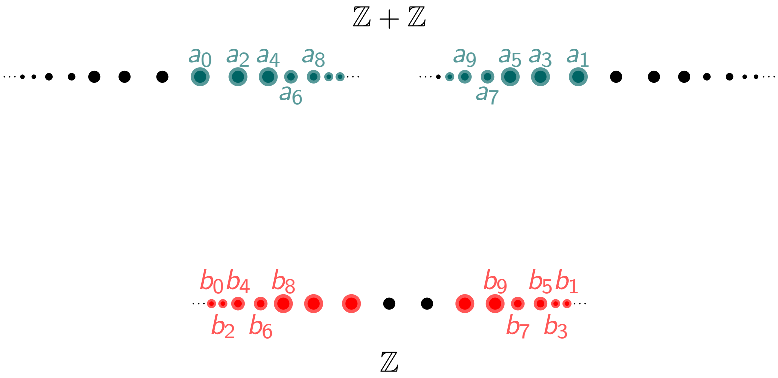

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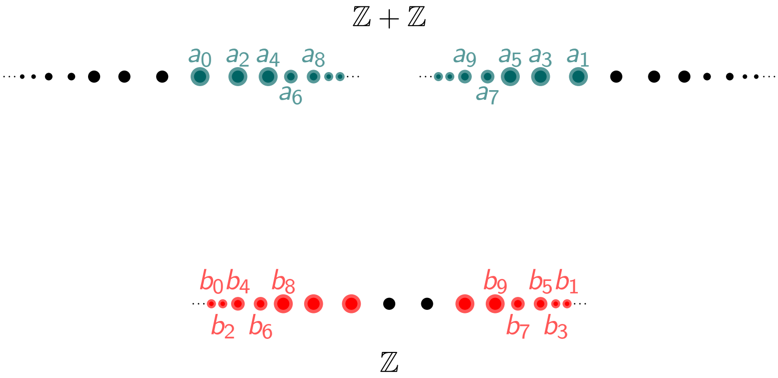
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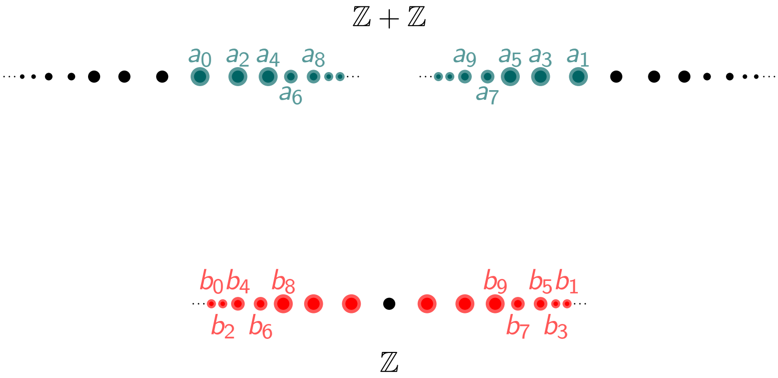

The long game

- The EF game $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$ of length ω captures similarity of structures $\mathfrak{A}$ and $\mathfrak{B}$ in a more complicated logic called $\mathcal{L}_{\infty\omega}$ where infinite conjunctions and disjunctions are allowed.

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- In particular, this sentence is in $\mathcal{L}_{\omega_1\omega}$ where only countably infinite conjunctions and disjunctions are allowed.

Lemma

If $\mathfrak{A}$ and $\mathfrak{B}$ are countable, then $\mathfrak{A} \cong \mathfrak{B}$ if and only if II *has a winning strategy in* $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$.

Proof.

Let a_i, $i \in \mathbb{N}$, enumerate $\mathfrak{A}$ and b_i, $i \in \mathbb{N}$, enumerate $\mathfrak{B}$. If II has a winning strategy, then on round i, I can just play a_i when i is even and b_i when i is odd, and the resulting function is an isomorphism $\mathfrak{A} \to \mathfrak{B}$. □

Tool: Ordinal Numbers

Definition

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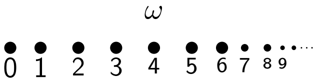
Lemma

$(X, <)$ *is a well-order if and only if there is no infinite decreasing sequence* $x_0 > x_1 > \dots$ *in* X.

- An ordinal number is a particularly nice representative of an isomorphism class of well-orders.
- An ordinal is well-ordered by the relation $\in$.
- Every well-order X is isomorphic to a unique ordinal, the *order-type* of X.
- One can do induction and recursion on ordinals.

Examples

- ω is the order type of $(\mathbb{N}, <)$:



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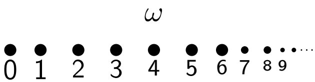

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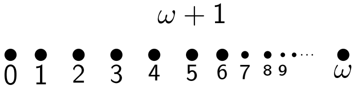
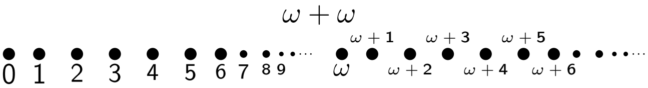


- $\omega + n$ is defined as one would expect
- $\omega + \omega$ (also known as $\omega \cdot 2$) is the order type of $\mathbb{N} + \mathbb{N}$:



Cardinal Numbers

- An ordinal κ is called a *cardinal number* if there are no $\alpha < \kappa$ such that there is a bijection $\alpha \to \kappa$.
- Examples: ω is a cardinal but $\omega + 1$ is not.
- The next cardinal after ω is ω_1, the first uncountable ordinal.
- If κ is a cardinal, we denote by κ^+ the least cardinal $> \kappa$.
- For every set X there is a unique cardinal κ such that there is a bijection $\kappa \to X$. Such κ is called the *cardinality of* X and denoted by $|X|$.

Example of Transfinite Recursion

Theorem

Every vector space has a basis.

Proof.

Let V be a vector space and let v_α, $\alpha < \kappa$, enumerate V, where $\kappa = |V|$. Then the set

$$\{v_\alpha \mid \alpha < \kappa, v_\alpha \notin \operatorname{span}(\{v_\beta \mid \beta < \alpha\})\}$$

is a basis of V. □

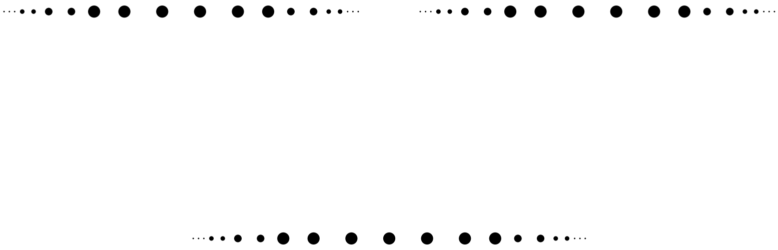
Dynamic EF Games

- A dynamic Ehrenfeucht–Fraïssé game is similar to the ordinary EF game, but it has an *ordinal clock* that ticks downwards.
- We denote by $\mathrm{EFD}_\alpha(\mathfrak{A}, \mathfrak{B})$ a game with clock α between the structures $\mathfrak{A}$ and $\mathfrak{B}$.
- Each round n I chooses some $\alpha_n < \alpha$ such that $\alpha_{n+1} < \alpha_n$ for every n. The game ends on the round n when I chooses $\alpha_n = 0$.

Example with clock $\omega + 2$

I plays:

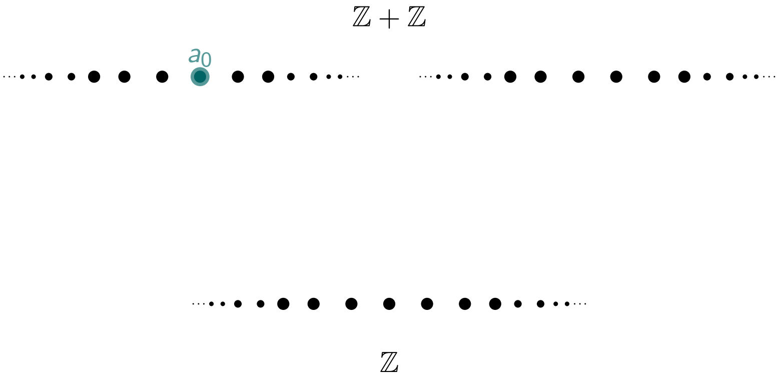
$\mathbb{Z} + \mathbb{Z}$



$\mathbb{Z}$

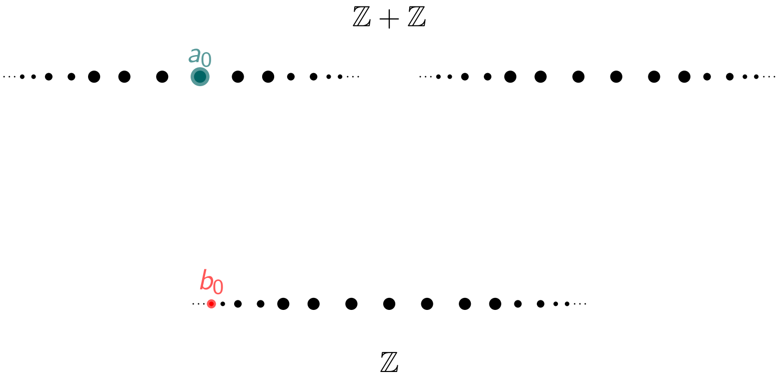
Example with clock $\omega + 2$

I plays: $\omega + 1$



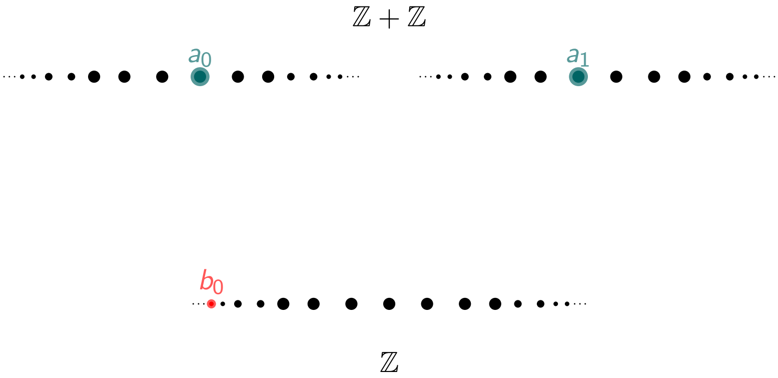
Example with clock $\omega + 2$

I plays: $\omega + 1$



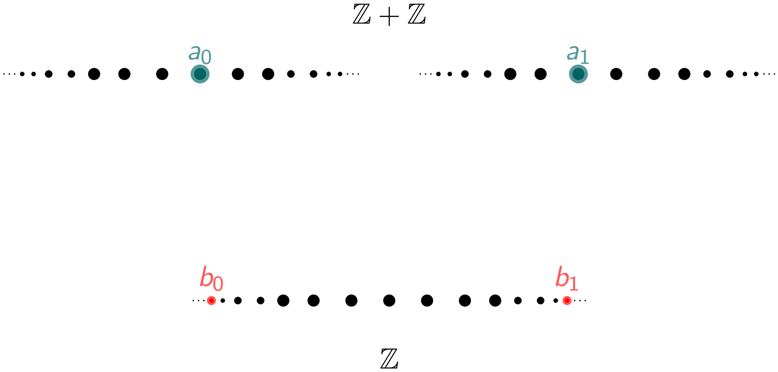
Example with clock $\omega + 2$

I plays: ω



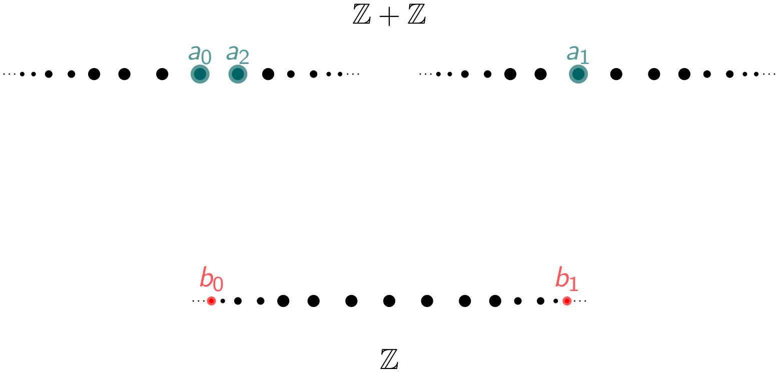
Example with clock $\omega + 2$

I plays: ω



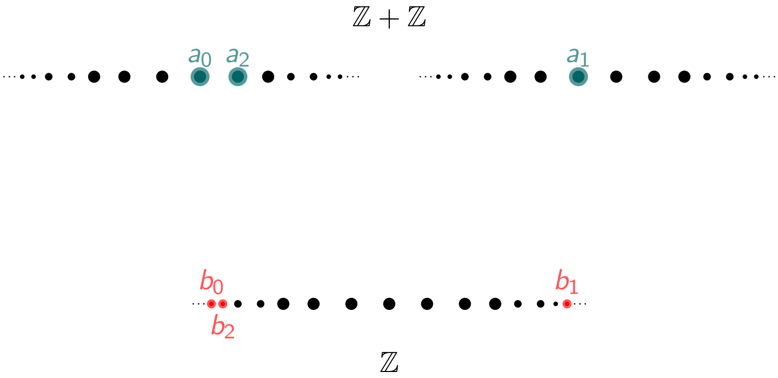
Example with clock $\omega + 2$

I plays: 13



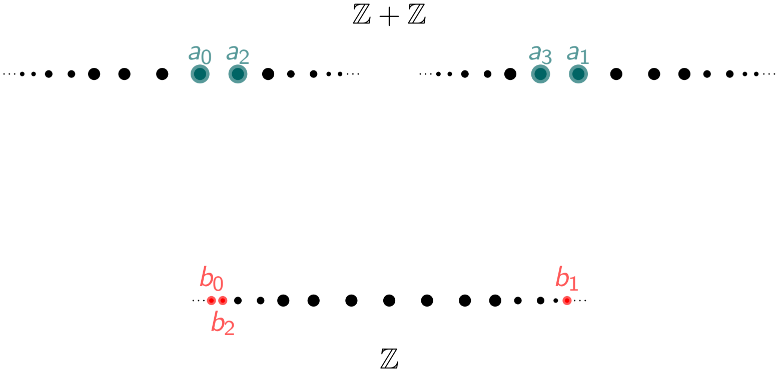
Example with clock $\omega + 2$

I plays: 13



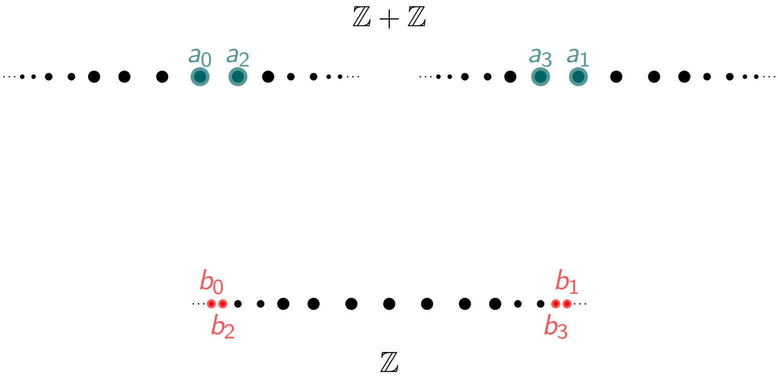
Example with clock $\omega + 2$

I plays: 12



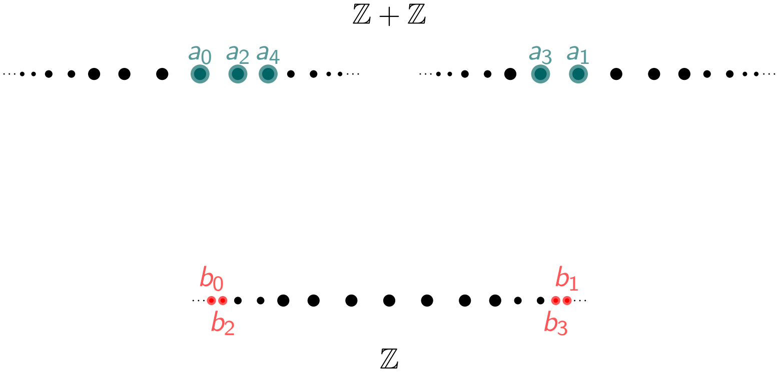
Example with clock $\omega + 2$

I plays: 12



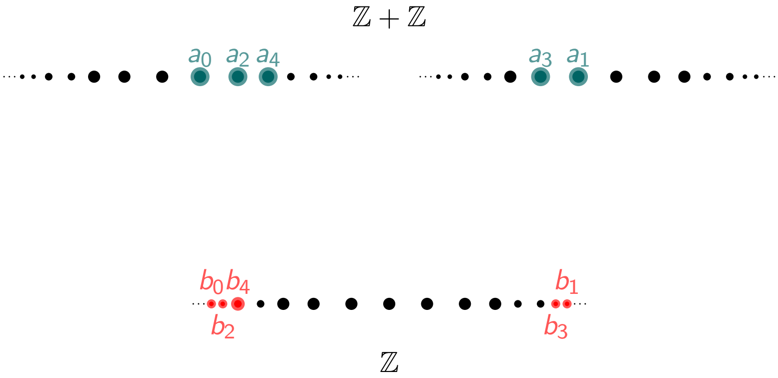
Example with clock $\omega + 2$

I plays: 11



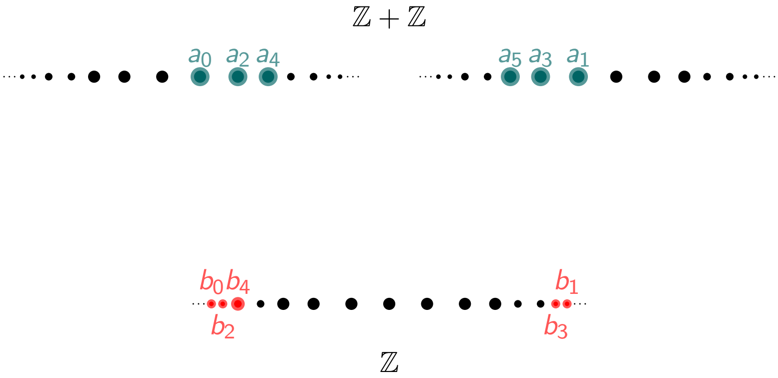
Example with clock $\omega + 2$

I plays: 11



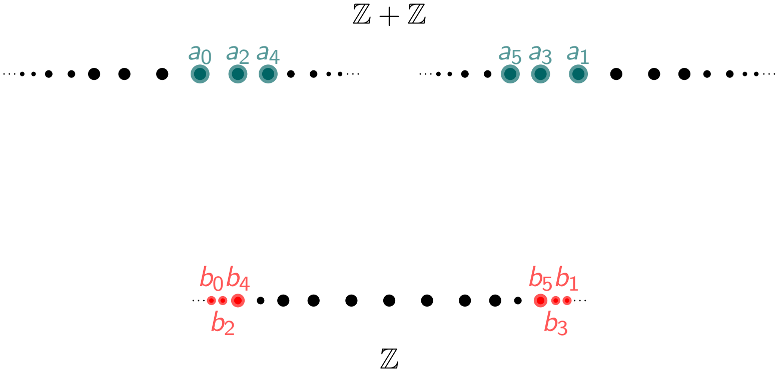
Example with clock $\omega + 2$

I plays: 10



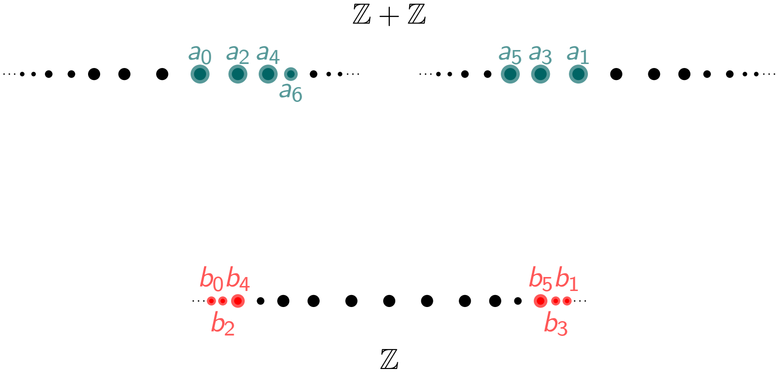
Example with clock $\omega + 2$

I plays: 10



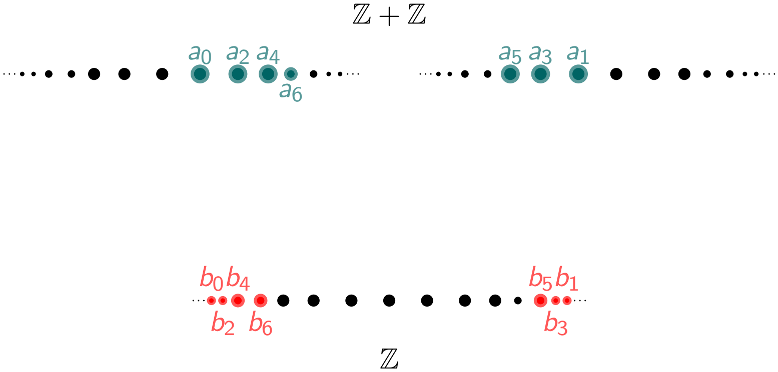
Example with clock $\omega + 2$

I plays: 9



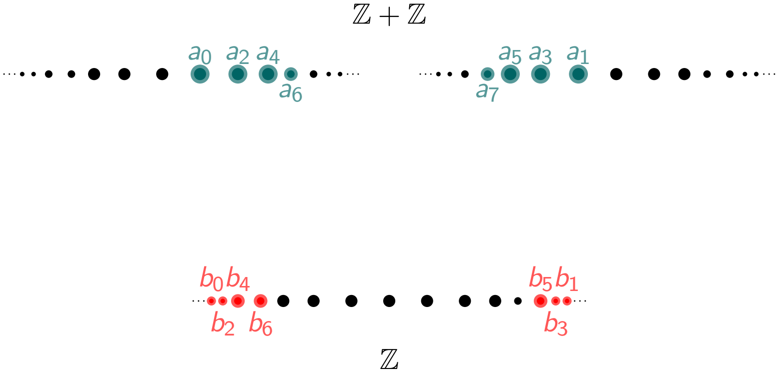
Example with clock $\omega + 2$

I plays: 9



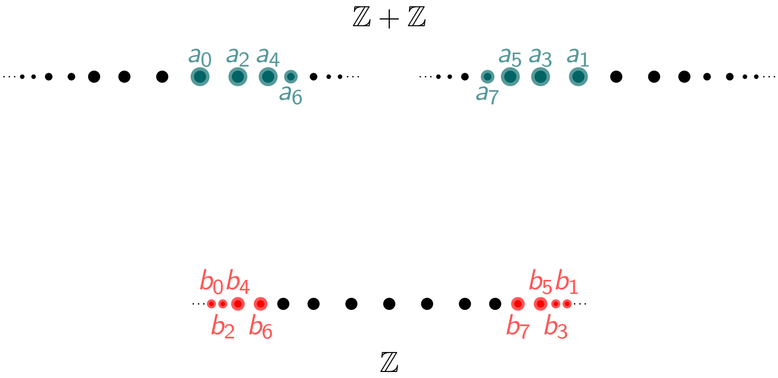
Example with clock $\omega + 2$

I plays: 8



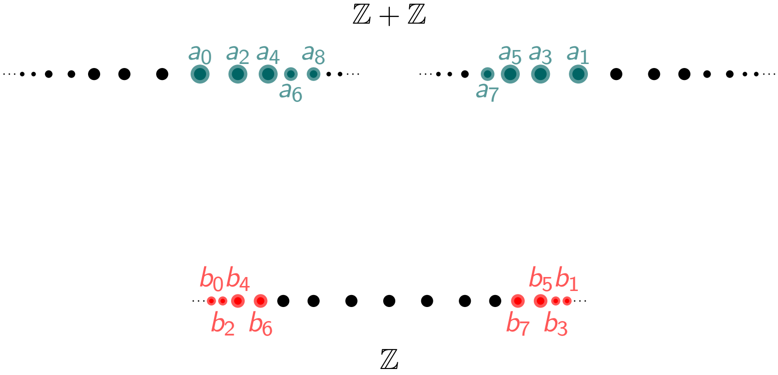
Example with clock $\omega + 2$

I plays: 8



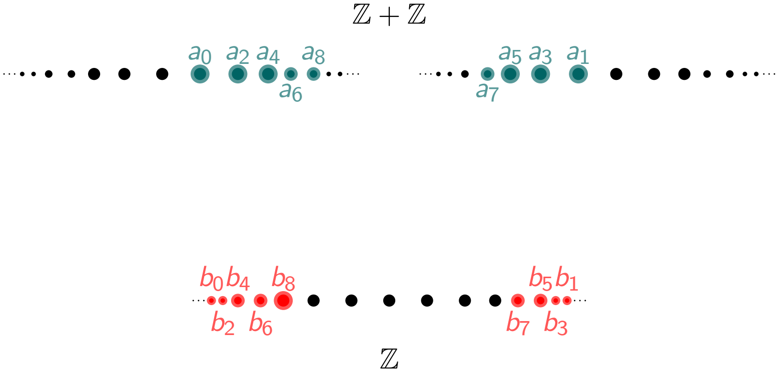
Example with clock $\omega + 2$

I plays: 7



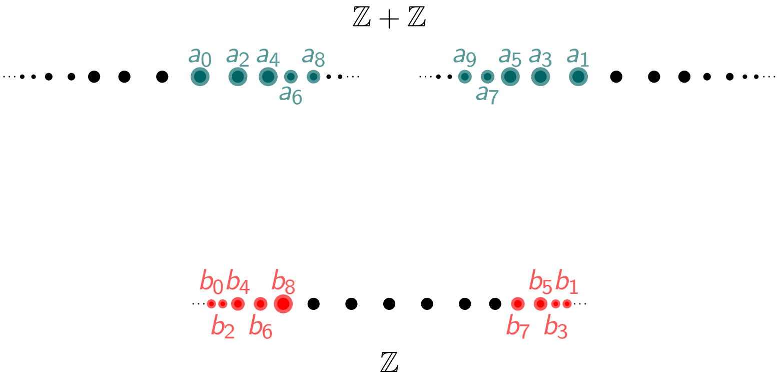
Example with clock $\omega + 2$

I plays: 7



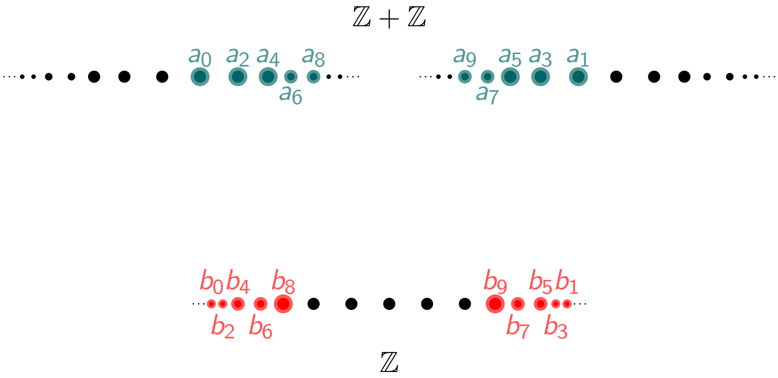
Example with clock $\omega + 2$

I plays: 6



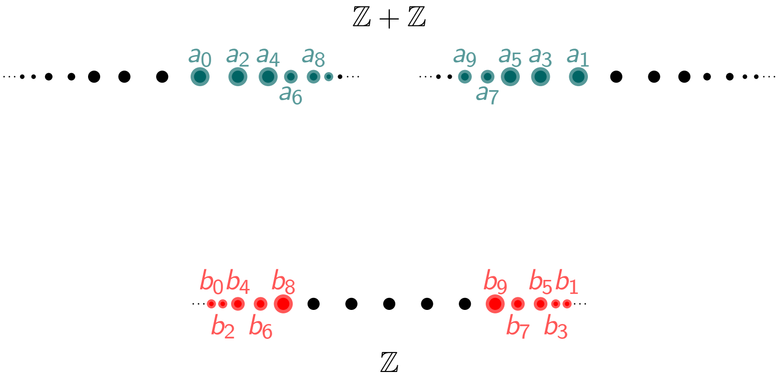
Example with clock $\omega + 2$

I plays: 6



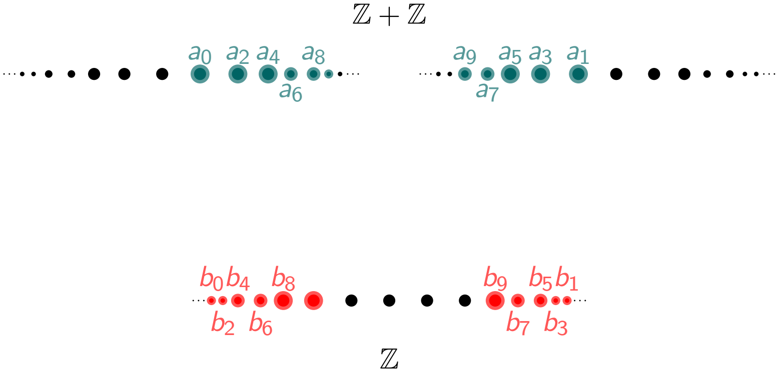
Example with clock $\omega + 2$

I plays: 5



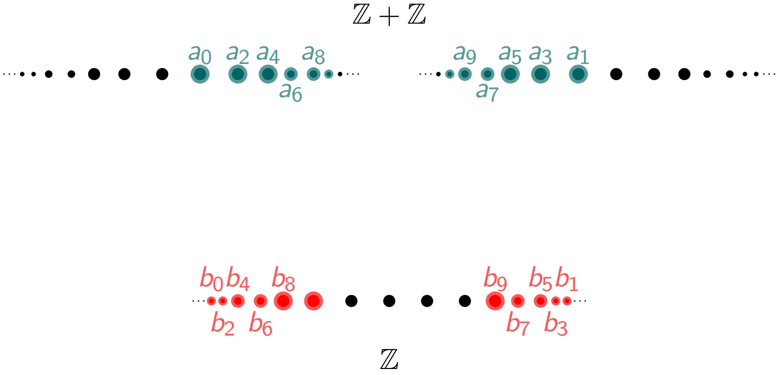
Example with clock $\omega + 2$

I plays: 5



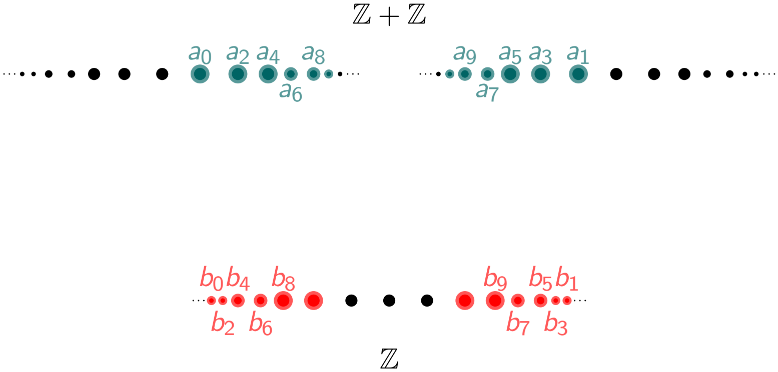
Example with clock $\omega + 2$

I plays: 4



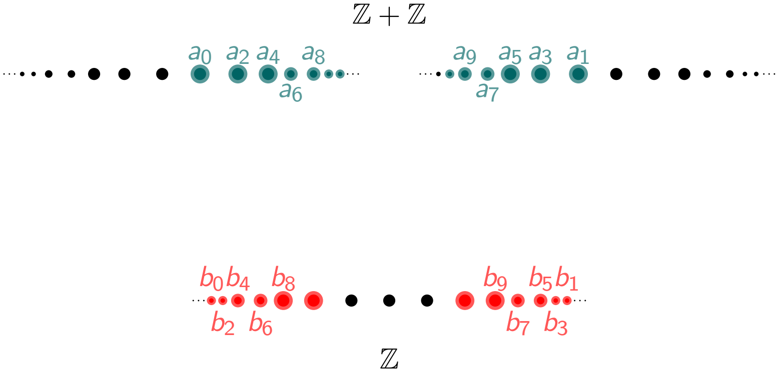
Example with clock $\omega + 2$

I plays: 4



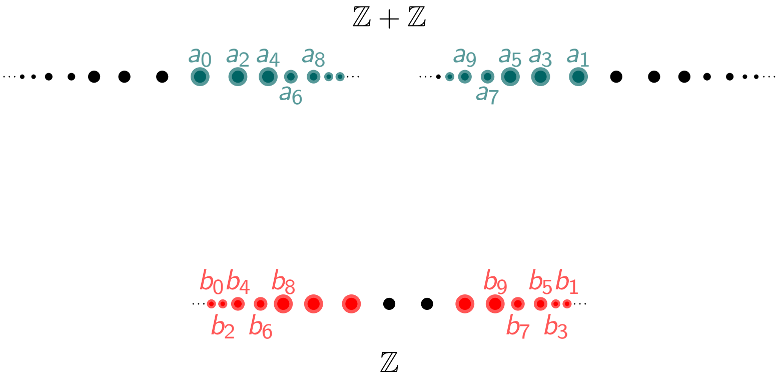
Example with clock $\omega + 2$

I plays: 3



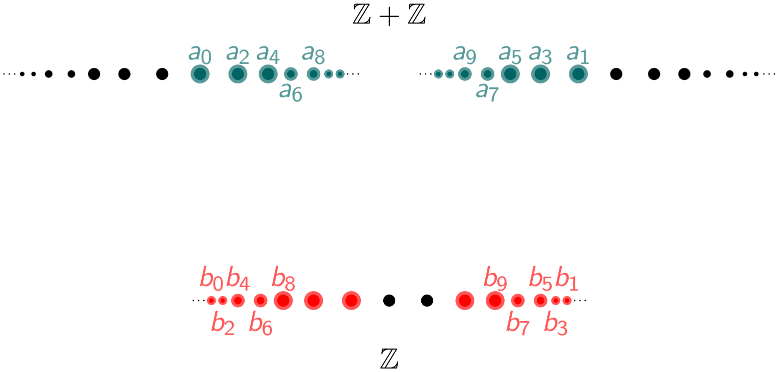
Example with clock $\omega + 2$

I plays: 3



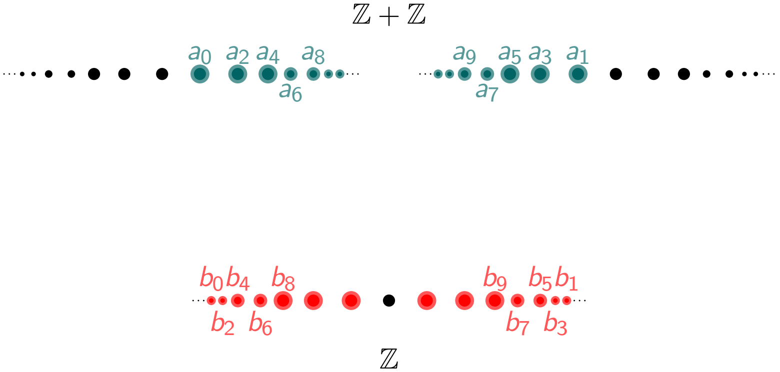
Example with clock $\omega + 2$

I plays: 2



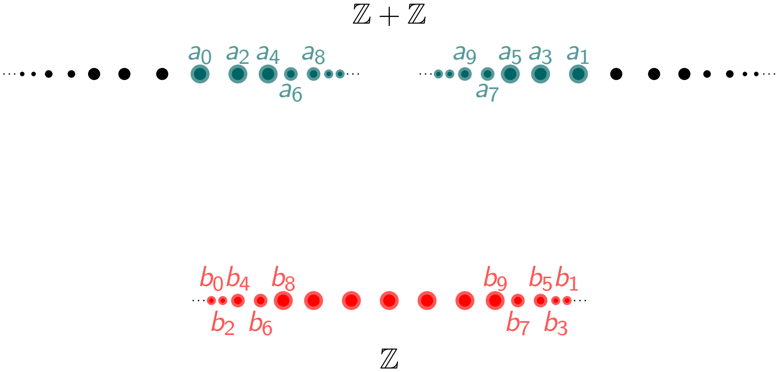
Example with clock $\omega + 2$

I plays: 2



Example with clock $\omega + 2$

I plays: 1



Dynamic Games vs. the Infinite Game

Theorem

Player II *has a winning strategy in* $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$ *if and only if she has a winning strategy in* $\mathrm{EFD}_\alpha(\mathfrak{A}, \mathfrak{B})$ *for every* $\alpha < (|\mathfrak{A}| + |\mathfrak{B}|)^+$.

For a proof, see again the book by Väänänen.

Dynamic Games vs. Logic

- For any structure $\mathfrak{A}$, there is a least ordinal α, called the *Scott rank of* $\mathfrak{A}$, such that whenever $\bar{a}, \bar{b} \in \mathfrak{A}^n$ and II has a winning strategy in

$$\mathrm{EFD}_\alpha((\mathfrak{A}, \bar{a}), (\mathfrak{A}, \bar{b})),$$

then also II has a winning strategy in

$$\mathrm{EFD}_{\alpha+1}((\mathfrak{A}, \bar{a}), (\mathfrak{A}, \bar{b})).$$

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- For every structure $\mathfrak{A}$, tuple $\bar{a} \in \mathfrak{A}^n$ and ordinal α, there is a formula $\varphi^{\bar{a}}_{\mathfrak{A}}(\bar{x})$ of $\mathcal{L}_{\infty\omega}$ such that for any other structure $\mathfrak{B}$ and $\bar{b} \in \mathfrak{B}^n$,

$$\mathfrak{B} \models \varphi^{\bar{a}}_{\mathfrak{A}}(\bar{b}) \iff \text{II has a winning strategy in } \mathrm{EFD}_\alpha((\mathfrak{A}, \bar{a}), (\mathfrak{B}, \bar{b})).$$

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- If $\alpha < \omega_1$, then $\varphi^{\bar{a}}_{\mathfrak{A}}(\bar{x}) \in \mathcal{L}_{\omega_1\omega}$.

Scott's Isomorphism Theorem

Theorem

For every countable $\mathfrak{A}$, there exists a sentence $\sigma_{\mathfrak{A}}$ of $\mathcal{L}_{\omega_1\omega}$ such that for any other countable $\mathfrak{B}$,

$$\mathfrak{B} \models \sigma_{\mathfrak{A}} \iff \mathfrak{B} \cong \mathfrak{A}.$$

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Proof sketch.

Using the formulas $\varphi^{\bar{a}}_{\mathfrak{A}}(\bar{x})$, construct a sentence $\sigma_{\mathfrak{A}}$ that expresses that "II can win enough dynamic games" (up until the Scott rank of $\mathfrak{A}$). Then $\mathfrak{B} \models \sigma_{\mathfrak{A}}$ iff II wins $\mathrm{EF}_\omega(\mathfrak{A}, \mathfrak{B})$. □

Approximate Games for Metric Structures

To summarize:

- Player II has a winning strategy in a game of infinite length between two *countable* structures if and only if the structures are isomorphic.
- The infinite game can be approximated by games of *dynamic length*.
- Dynamic games correspond to formulas of a certain logic.
- Thus we can describe a countable structure up to isomorphism using said logic.

Metric structures

For metric structures, such as

- metric spaces,
- Banach spaces, and
- Hilbert spaces,

we want

- an infinite game such that two *separable* structures are isomorphic if and only if II has a winning strategy,
- dynamic games for approximating the infinite game, and
- formulas corresponding the dynamic games.

Linear Isomorphisms of Banach Spaces

- If $\mathfrak{A}$ and $\mathfrak{B}$ are Banach spaces, a bijection $f : \mathfrak{A} \to \mathfrak{B}$ is a linear isomorphism if it is linear and bi-Lipschitz.
- For $\varepsilon \geq 0$, if f is a linear e^{ε}-bi-Lipschitz function, then we call f an ε-isomorphism.

- Consider as atomic formulas expressions

$$\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \leq 1 \quad \text{and} \quad \left\|\sum_{i=0}^{n-1} c_i x_i\right\| \geq 1,$$

where $c_i \in \mathbb{Q}$.

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where $c_i \in \mathbb{Q}$.

- An atomic formula $\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \leq 1$ is k-good (for $k \in \mathbb{N}$) if $n = k$ and $|c_i| \leq k$ for all $i < n$, and similarly for $\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \geq 1$.

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$$\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \leq 1 \quad \text{and} \quad \left\|\sum_{i=0}^{n-1} c_i x_i\right\| \geq 1,$$
where $c_i \in \mathbb{Q}$.
- An atomic formula $\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \leq 1$ is k-good (for $k \in \mathbb{N}$) if $n = k$ and $|c_i| \leq k$ for all $i < n$, and similarly for $\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \geq 1$.
- We define the ε-approximation of $\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \leq 1$ (for $\varepsilon \geq 0$) to be
$$\left\|e^{-\varepsilon} \sum_{i=0}^{n-1} c_i x_i\right\| \leq 1$$
and of $\left\|\sum_{i=0}^{n-1} c_i x_i\right\| \geq 1$ to be
$$\left\|e^{\varepsilon} \sum_{i=0}^{n-1} c_i x_i\right\| \geq 1.$$
We denote by $\mathrm{Appr}(\varphi, \varepsilon)$ the ε-approximation of φ.

Approximate Games

- If $\mathfrak{A}$ and $\mathfrak{B}$ are two Banach spaces and $D_{\mathfrak{A}} \subseteq \mathfrak{A}$ and $D_{\mathfrak{B}} \subseteq \mathfrak{B}$ are dense sets and $\varepsilon \geq 0$, then the game

$$\mathrm{EF}^{\mathfrak{A},\mathfrak{B}}_{\omega,\varepsilon}(D_{\mathfrak{A}}, D_{\mathfrak{B}})$$

is played like the ordinary EF game between two sets but in addition in each round i, I picks some $\varepsilon_i > \varepsilon$.

- In the end II wins if for every $k \in \mathbb{N}$ and k-good formula $\varphi(x_0, \ldots, x_{k-1})$,

$$\mathfrak{A} \models \varphi(a_{i_0}, \ldots, a_{i_{k-1}}) \implies \mathfrak{B} \models \mathrm{Appr}(\varphi, \varepsilon_k)(b_{i_0}, \ldots, b_{i_{k-1}})$$

for all $i_0, \ldots, i_{k-1} \geq k$.

Theorem (Hirvonen–P.)

Let $\mathfrak{A}$ *and* $\mathfrak{B}$ *be separable Banach spaces and* $\varepsilon \geq 0$*. Then* II *has a winning strategy in* $\mathrm{EF}^{\mathfrak{A},\mathfrak{B}}_{\omega,\varepsilon}(\mathfrak{A},\mathfrak{B})$ *if and only if there exists an* ε*-isomorphism* $\mathfrak{A} \to \mathfrak{B}$.

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Let $\mathfrak{A}$ *and* $\mathfrak{B}$ *be separable Banach spaces and* $\varepsilon \geq 0$. *Then* II *has a winning strategy in* $\mathrm{EF}^{\mathfrak{A},\mathfrak{B}}_{\omega,\varepsilon}(\mathfrak{A},\mathfrak{B})$ *if and only if there exists an* ε*-isomorphism* $\mathfrak{A} \to \mathfrak{B}$.

Proof sketch.

If II has a winning strategy, let $D_{\mathfrak{A}} \subseteq \mathfrak{A}$ and $D_{\mathfrak{B}} \subseteq \mathfrak{B}$ be countable dense sets. We let I play all the elements of both dense sets as his moves in the infinite game. Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ denote the elements played in each space. Now one can prove that if $(i_k)_{k\in\mathbb{N}}$ is an increasing sequence of indices, then $(a_{i_k})_{k\in\mathbb{N}}$ is Cauchy iff $(b_{i_k})_{k\in\mathbb{N}}$ is Cauchy. Then the mapping that maps limit points of $(a_i)_{i\in\mathbb{N}}$ to the limit points of $(b_i)_{i\in\mathbb{N}}$ is an ε-isomorphism. □

Dynamic Games and Formulas

- We can define dynamic ε-games corresponding to the infinite game and get similar results.
- Similarly to the classical case, we can build formulas corresponding to the dynamic games.
- We get an ε-Scott sentence of a Banach space $\mathfrak{A}$. Interestingly, this sentence might not be an element of $\mathcal{L}_{\omega_1\omega}$, i.e. it could contain uncountable conjunctions or disjunctions.
- However, it is an element of $\mathcal{L}_{\omega_2\omega}$ (so the conjunctions and disjunctions are not *that long*).
- Can't be bothered with the details, see the paper if you're somehow still interested. ;)

References for the Curious

Itaï Ben Yaacov, Michal Doucha, André Nies, and Todor Tsankov.
Metric Scott analysis.
Adv. Math., 318:46–87, 2017.

C. Henson.
Nonstandard hulls of Banach spaces.
Israel Journal of Mathematics, 25:108–144, 1976.

Åsa Hirvonen and Joni Puljujärvi.
Games and Scott sentences for positive distances between metric structures.
arXiv e-prints, page arXiv:2102.00993, June 2021.

Jouko Väänänen.
Models and games, volume 132 of *Cambridge Studies in Advanced Mathematics*.
Cambridge University Press, Cambridge, 2011.