Invariant measures: What and why

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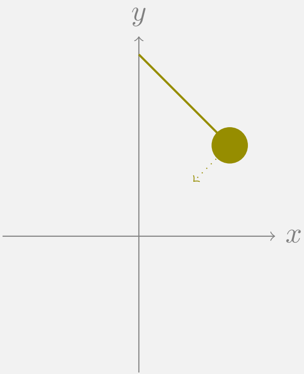
Outline

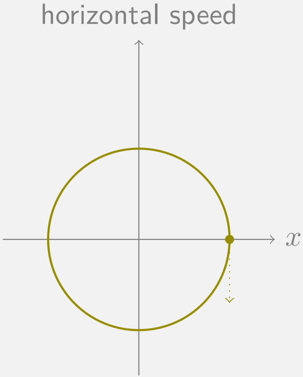
1. Finite dimensions
   - State space and ODEs
   - Invariant measure
   - Recurrence theorem
2. Nonlinear Schrödinger equation
   - Solution via Fourier series
   - The problem statement
3. Random initial data
   - Intuitive construction
   - Sobolev spaces
4. Invariant measure for NLS
   - Truncation trick
   - The approximation
   - Improved regularity theory
   - Recurrence theorem

Part 1: Finite dimensions

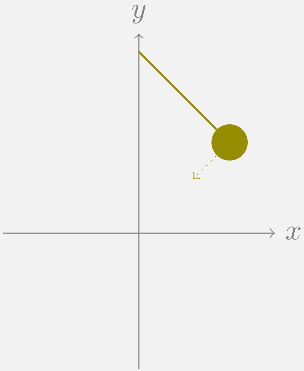
Basic definitions

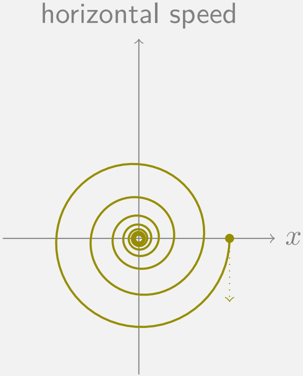
State space: pendulum



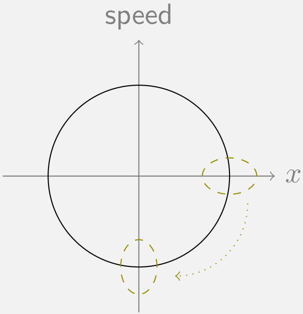


State space: pendulum with friction

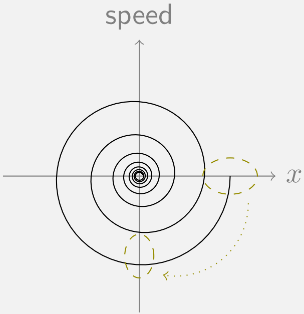




Invariant measure



No friction: invariant



Friction: not invariant

Same formally

Definition

Let $X \subset \mathbb{R}^d$ be the *state space* of the system, and μ a measure on it. A *flow* $\Phi_t(x)\colon X \times \mathbb{R} \to X$ is μ-invariant if

$$\mu(\Phi_t(A)) = \mu(A) \quad \text{for all measurable } A \subset X \text{ and } t \in \mathbb{R}.$$

We assume Φ to be bijective and continuous.

Recurrence theorem

Theorem (Poincaré)

Assume that $\mu(X) < \infty$*. If* $\mu(A) > 0$*, then there exist infinitely many* $t \in \mathbb{R}$ *such that*

$$\mu\left(\Phi_t(A) \cap A\right) > 0.$$

That is, the flow returns to any region of initial data.

Proof.

If there wasn't, then we'd have

$$\mu\left(\bigcup_{t\in\mathbb{N}} \Phi_t(A)\right) = \sum_{t\in\mathbb{N}} \mu(\Phi_t(A)) = \infty,$$

a contradiction. Repeat with $\Phi_t(A)$ as initial data. □

How to find invariant measures

Theorem (Liouville)

Let Φ be a Hamiltonian flow on $\mathbb{R}^d$. Then Φ is invariant w.r.t. Lebesgue $\mathrm{d}x$ on $\mathbb{R}^d$.

Hamiltonian H

H corresponds to the "total energy" of the system and is conserved by Φ. Then $\mu(x) = f(H(x))\,\mathrm{d}x$ is invariant:

$$\mu(\Phi_t(A)) = \int_{\Phi_t(A)} f(H(x))\,\mathrm{d}x = \int_A f(H(\Phi_t(x)))\,\mathrm{d}x = \int_A f(H(x))\,\mathrm{d}x = \mu(A).$$

Part 2: Nonlinear Schrödinger equation

A wave-like PDE, and some Fourier analysis

The protagonist

Nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u(x,t) + \partial_{xx} u(x,t) = \pm \left| u(x,t) \right|^2 u(x,t), \\ u(x,0) = u_0(x), \end{cases}$$

where u_0 is a complex function on $\mathbb{T} = [0, 2\pi]$.

- Complex-valued
- Describes waves in e.g. Bose–Einstein condensate
- Dispersive: speed of sound depends on frequency
- $\pm$ causes or prevents blow-up

Solving linear Schrödinger

$$i\partial_t u(x,t) + \partial_{xx} u(x,t) = 0.$$

Take Fourier transform in space and time:

$$i(i\vartheta)\hat{u}(k,\vartheta) + (ik)^2 \hat{u}(k,\vartheta) = 0.$$

Combine common factors:

$$(\vartheta + k^2)\hat{u}(k,\vartheta) = 0.$$

This implies $\operatorname{supp}\hat{u} \subset \{(k,-k^2)\}$. So $\hat{u}$ depends on k only $\implies$ same as $\hat{u_0}$.
Inverse transform:

$$u(x,t) = \sum_{k\in\mathbb{Z}} e^{ikx-ik^2t}\hat{u_0}(k) =: e^{it\Delta}u_0(x).$$

Solving nonlinear Schrödinger

$$\begin{cases} i\partial_t u(x,t) + \partial_{xx} u(x,t) = \pm \left|u(x,t)\right|^2 u(x,t), \\ u(x,0) = u_0(x). \end{cases}$$

Linear case solved by $e^{it\Delta}u_0$.
Nonlinear case by Duhamel principle:

$$u(x,t) = e^{it\Delta}u_0 \pm \int_0^t e^{i(t-s)\Delta} \left|u(x,t)\right|^2 u(x,t)\,\mathrm{d}s.$$

- Fixed-point iteration converges in suitable space, here $L^4(\mathbb{T} \times [0,\tau])$
- Only valid up to time τ (small)
- Details are awful (major opening in analytic number theory)

Global solution

Theorem (Local solution)

Given $u_0 \in L^2(\mathbb{T})$*, unique solution* $u(x,t)$ *up to time* $\tau \simeq \|u_0\|_2^{-C}$*.*

Theorem (Conservation of mass)

If u *solves NLS, then* $\|u(\cdot,t)\|_2 = \|u_0\|_2$ *for all* $t \in \mathbb{R}$*.*

Corollary

Given $u_0 \in L^2(\mathbb{T})$*, there is unique* $u(x,t)$ *for all* $t \in \mathbb{R}$*.*

Conservation laws, I

Theorem (Conservation of mass)

If u solves NLS, then $\|u(\,\cdot,t)\|_2 = \|u_0\|_2$ for all $t \in \mathbb{R}$.

Proof.

$$\partial_t \|u(\cdot,t)\|_2^2 = \int_{\mathbb{T}} \partial_t \left[u(x,t)\overline{u(x,t)} \right] \mathrm{d}x,$$

use product rule and solve $\partial_t u$ from NLS. Everything cancels out. □

Conservation laws, II

Theorem (Conservation of energy)

If u solves NLS, then its Hamiltonian satisfies $H(u(\cdot,t)) = H(u(\cdot,0))$ for all $t \in \mathbb{R}$.

Invariant measure for NLS

- Initial data $u_0 \sim \mathcal{D}$, where $\mathcal{D}$ some random distribution (how?)
- Want $u_t \sim \mathcal{D}$ for all $t \in \mathbb{R}$
- Two conservation laws at our disposal
- If Liouville did hold in infinite dimensions, we'd have

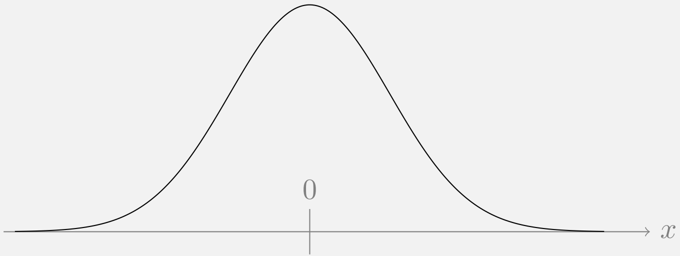
$$f(H(u,t))\,\mathrm{d}x \quad \text{invariant}$$

- But a Lebesgue measure $\mathrm{d}x$ cannot exist on $\mathbb{C}^{\mathbb{Z}}$!

Part 3: Random initial data

Measures on function spaces, and a bit of Sobolev

Normal distribution



- Real normal distribution $\mathcal{N}(\text{mean}, \text{variance})$
- Complex: $X + iY$ where $X, Y \sim \mathcal{N}(0, \text{variance}/2)$ independent
- Linear combination of normal distributions still normal

Idea for our random functions

Fourier series with random coefficients

$$\hat{u_0}(k) \sim \mathcal{N}(0, 1/k^{?})$$

so that the initial function is

$$u_0(x) = \sum_{k \in \mathbb{Z}} \frac{e^{ikx}}{k^{?}} g_k,$$

where g_k independent complex normal with variance 1.

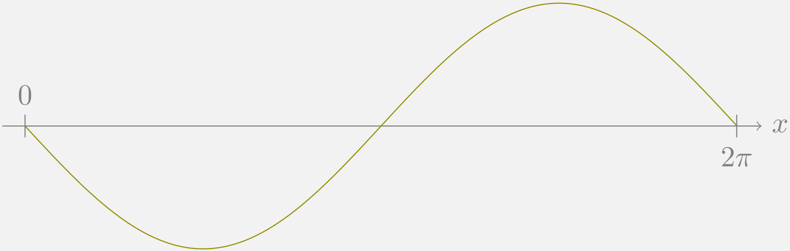
Convergence

- Need $?$ such that the series converges (almost everywhere, almost surely)
- Also do something about $k = 0$

Simplified example

$$f(x) = \sum_{k=1}^{N} \frac{g_k}{k} \sin(kx)$$

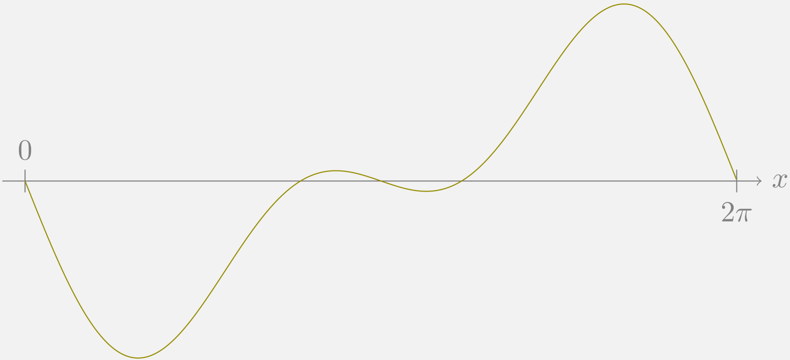
$$N = 1$$



Simplified example

$$f(x) = \sum_{k=1}^{N} \frac{g_k}{k} \sin(kx)$$

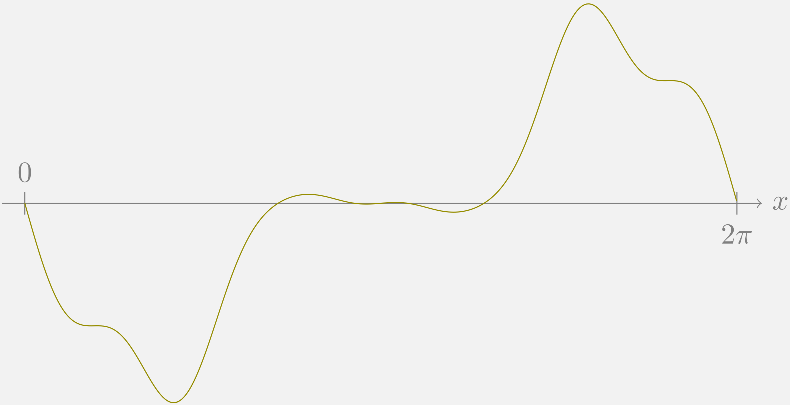
$$N = 2$$



Simplified example

$$f(x) = \sum_{k=1}^{N} \frac{g_k}{k} \sin(kx)$$

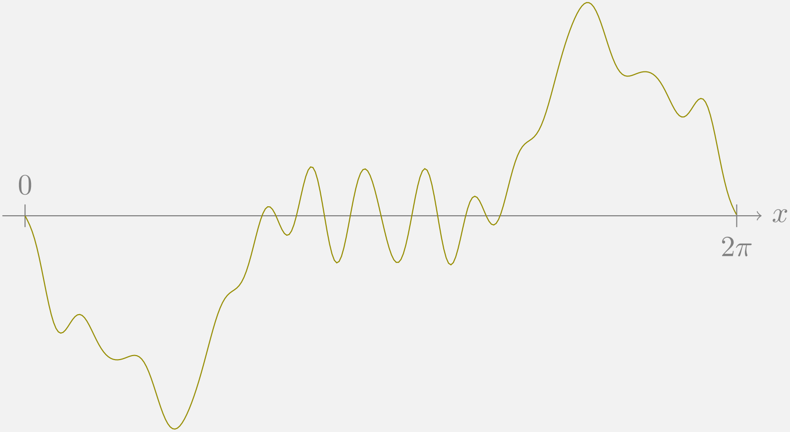
$$N = 8$$



Simplified example

$$f(x) = \sum_{k=1}^{N} \frac{g_k}{k} \sin(kx)$$

$$N = 20$$



Convergence of series

Theorem (Khintchine inequality)

Let (b_k) be a fixed sequence and (g_k) complex Gaussian. Then

$$\mathbb{E}\left[\sum_{k\in\mathbb{Z}} b_k g_k\right]^q \lesssim q^{q/2}\,\|(b_k)\|_{\ell^2}^q\,, \qquad 1 \leq q < \infty.$$

Our series

We have $\hat{u_0}(k) \sim \mathcal{N}(0, 1/k^\alpha)$, and by choosing $q = 2$ in the above we get

$$\mathbb{E}\,\|u_0\|_2^2 = \mathbb{E}\left\|\sum_{k\in\mathbb{Z}} \frac{e^{ikx}}{k^\alpha} g_k\right\|_2^2 \lesssim \sum_{k\in\mathbb{Z}} \left|\frac{e^{ikx}}{k^\alpha}\right|^2 .$$

So $u_0 \in L^2$ almost surely if $2\alpha > 1$.

Fractional Sobolev spaces

Idea

Usual Sobolev norm defined on $\mathbb{T}$ by

$$\|u\|_{H^1} = \left[\|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2\right]^{1/2}.$$

But by Fourier transform

$$\|u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2 = \sum_{k\in\mathbb{Z}} |\hat{u}(k)|^2 + |(ik)\hat{u}(k)|^2 = \sum_{k\in\mathbb{Z}} (1+k^2)\,|\hat{u}(k)|^2\,.$$

Definition

Function u belongs to H^s if $(1+k^2)^{s/2}\,\hat{u}(k)$ defines an L^2 function.

A bit more regularity

Random series in H^s

Put $g_k \sim \mathcal{N}(0, 1/k)$. Fractional Sobolev norm

$$\mathbb{E}\, \|u_0\|_{H^s}^2 = \mathbb{E} \left\| \sum_{k \in \mathbb{Z}} (1+k^2)^{s/2} \frac{e^{ikx}}{k} g_k \right\|_2^2 \lesssim \sum_{k \in \mathbb{Z}} \left| \frac{e^{ikx}}{k^{1-s}} \right|^2 .$$

So $u_0 \in H^s$ almost surely if $2 - 2s > 1 \implies s < 1/2$.

Intuition: u_0 has "almost half a derivative"

- We just defined a *Brownian bridge* on $\mathbb{T}$
- Remark: we're still glossing over $k = 0$

Part 4: Invariant measure for NLS

Putting it all together

What we've seen so far

Problem: Find $\mathcal{D}$ such that

$$\begin{cases} i\partial_t u(x,t) + \partial_{xx} u(x,t) = \pm \left|u(x,t)\right|^2 u(x,t), \\ u(x,t) \sim \mathcal{D} \quad \text{for all } t \in \mathbb{R}. \end{cases}$$

Conservation laws

- L^2 norm and Hamiltonian conserved
- $f(H(\,\cdot\,))\,\mathrm{d}x$ invariant if finite dimension (Liouville)

Gaussian measures

$$u_0(x) = \sum_{k\in\mathbb{Z}} e^{ikx}\frac{g_k}{k} \quad \text{is a random function of regularity } H^{1/2-\varepsilon}$$

Why Gaussian measure?

Physics, no details

Finite-dimensional Hamiltonian system has Gibbs measure $\mathrm{d}\mu(x) = \exp(-\beta H(x))\,\mathrm{d}x$.
For NLS the Hamiltonian is

$$H(u) = \mp\frac{1}{p}\left\|u\right\|_4^4 + \frac{1}{2}\left\|\partial_x u\right\|_2^2 .$$

Warning: hand-waving!

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$$H(u) = \mp\frac{1}{p}\,\|u\|_4^4 + \frac{1}{2}\,\|\partial_x u\|_2^2\,.$$

Warning: hand-waving!

Infinitely many Fourier coefficients

$$\mathrm{d}\mu(u) = \exp\left(\pm\frac{\beta}{p}\,\|u\|_4^4\right)\exp\left(-\frac{\beta}{2}\,\|\partial_x u\|_2^2\right)\prod_{k\in\mathbb{Z}}\mathrm{d}\hat{u}(k).$$

Why Gaussian measure?

Physics, no details

Finite-dimensional Hamiltonian system has Gibbs measure $\mathrm{d}\mu(x) = \exp(-\beta H(x))\,\mathrm{d}x$.
For NLS the Hamiltonian is

$$H(u) = \mp\frac{1}{p}\,\|u\|_4^4 + \frac{1}{2}\,\|\partial_x u\|_2^2\,.$$

Warning: hand-waving!

Move the first term to Fourier space:

$$\mathrm{d}\mu(u) = \exp\left(\pm\frac{\beta}{p}\,\|u\|_4^4\right)\exp\left(-\frac{\beta}{2}\sum_{k\in\mathbb{Z}}|(ik)\hat{u}(k)|^2\right)\prod_{k\in\mathbb{Z}}\mathrm{d}\hat{u}(k).$$

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Finite-dimensional Hamiltonian system has Gibbs measure $\mathrm{d}\mu(x) = \exp(-\beta H(x))\,\mathrm{d}x$.
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$$H(u) = \mp\frac{1}{p}\left\|u\right\|_4^4 + \frac{1}{2}\left\|\partial_x u\right\|_2^2 .$$

Warning: hand-waving!

A wild Gaussian measure appears!

$$\mathrm{d}\mu(u) = \exp\left(\pm\frac{\beta}{p}\left\|u\right\|_4^4\right)\prod_{k\in\mathbb{Z}}\exp\left(-\frac{\beta\left|\hat{u}(k)\right|^2}{2(1/k)}\right)\mathrm{d}\hat{u}(k).$$

Why Gaussian measure?

Physics, no details

Finite-dimensional Hamiltonian system has Gibbs measure $\mathrm{d}\mu(x) = \exp(-\beta H(x))\,\mathrm{d}x$.
For NLS the Hamiltonian is

$$H(u) = \mp\frac{1}{p}\|u\|_4^4 + \frac{1}{2}\|\partial_x u\|_2^2 .$$

Warning: hand-waving!

Let's just ignore $k = 0$ (actually you choose Lebesgue on an interval):

$$\mathrm{d}\mu(u) = \exp\left(\pm\frac{\beta}{p}\|u\|_4^4\right)\prod_{k\neq 0}\exp\left(-\frac{\beta\,|\hat{u}(k)|^2}{2(1/k)}\right)\mathrm{d}\hat{u}(k).$$

Some approximation required

Truncated measure

Define a measure on $\mathbb{C}^{2N}$ by

$$\mathrm{d}\mu_N(u) = \exp\left(\pm\frac{\beta}{p}\,\|P_N u\|_4^4\right) \prod_{\substack{k\neq 0\\ |k|\leq N}} \exp\left(-\frac{\beta\,|\hat{u}(k)|^2}{2(1/k)}\right) \mathrm{d}\hat{u}(k),$$

where P_N projection to frequencies $|k| \leq N$. Liouville $\implies$ invariant!

Approximation

- Extend this to $\mathbb{C}^{\mathbb{Z}\setminus\{0\}}$ with (unweighted) Gaussian part
- Uniformly bounded in N $\implies$ convergence to μ
- Details technical (measure bound + bounds for norm growth in time)

Some approximation required

Truncated measure

Define a measure on $\mathbb{C}^{2N}$ by

$$\mathrm{d}\mu_N(u) = \exp\left(\pm\frac{\beta}{p}\|P_N u\|_4^4\right)\prod_{\substack{k\neq 0\\ |k|\leq N}}\exp\left(-\frac{\beta\,|\hat{u}(k)|^2}{2(1/k)}\right)\mathrm{d}\hat{u}(k),$$

where P_N projection to frequencies $|k| \leq N$. Liouville $\implies$ invariant!

Invariance

Split $u_0 = P_N u_0 + P_{>N} u_0$ where $P_N u_0 \sim \mu_N$ and $P_{>N} u_0 \sim$ Gaussian, both parts invariant under truncated equation

$$i\partial_t u(x,t) + \partial_{xx} u(x,t) = \pm P_N\Big[\,|P_N u(x,t)|^2\, P_N u(x,t)\Big].$$

Why would you do this? I

Improved regularity

Let $u_0 \in H^{1/2-\varepsilon}$. Then there is a $H^{1/2-\varepsilon}$ solution to NLS with this initial data almost surely w.r.t. the measure.

Take a neighbourhood of u_0, and you get solutions.

Growth bound

As a byproduct, for any $\delta > 0$ there is set of probability $1 - \delta$ where

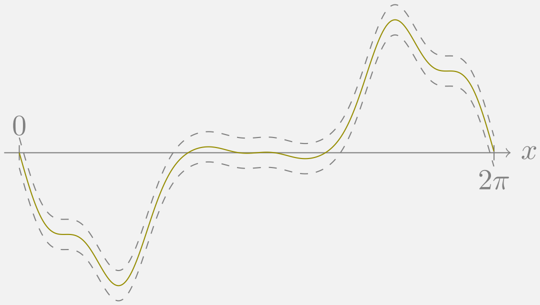
$$\|v(\cdot, t)\|_{H^{1/2-\varepsilon}} \lesssim 1 + \sqrt{\log\left(\frac{1+|t|}{\delta}\right)}.$$

Why would you do this? II

Theorem (Poincaré)

If $\mu(A) > 0$*, then there exist infinitely many* $t \in \mathbb{R}$ *such that* $\mu\left(\Phi_t(A) \cap A\right) > 0$*.*

Fermi–Pasta–Ulam–Tsingou paradox: complicated systems are almost periodic!



Further questions

$$\begin{cases} i\partial_t u(x,t) + \partial_{xx} u(x,t) = \pm \left|u(x,t)\right|^2 u(x,t), \\ u(x,0) = u_0(x), \quad x \in \mathbb{T} \end{cases}$$

What about...

- dimension greater than 1?
- nonlinearity with higher power?
- fixed potential, or stochastic forcing?
- infinite volume?
- other equations?

You made it to the end!

This talk was based on

Laarne: *Periodic nonlinear Schrödinger equation* (MSc thesis, 2021)