Fractional Gaussian fields and the Gaussian multiplicative chaos measure

Patrik Nummi

University of Helsinki

DOMAST Seminar
March 11, 2022

Outline of the talk

1. Introduction
2. Random fields
   - Some basics on random processes and fields
   - Gaussians: on finite- and infinite-dimensional spaces
   - Examples of Gaussian processes
3. Fractional fields
   - Construction of the white noise L^2 space
4. Gaussian multiplicative chaos measure
   - Definition and a sketch of how to construct GMC
   - Basic properties of GMC

Motivation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- The *stochastic pressure equation*: $U \subset \mathbb{R}^n$ a bounded, smooth domain, consider

$$-\nabla_x \cdot (p(x;\omega) \diamond \nabla_x u(x;\omega)) = 0, \qquad x \in U, \, \omega \in \Omega,$$

  with some (non-random) boundary data. Here $\diamond$ is the so-called *Wick product*; acts as a stochastic renormalization of sorts
- Related to the ST1 Deep Heat project in Espoo, models water seeping through a porous medium. The plot of the solution looks like a percolation diagram
- Modelling suggests to take $p(x;\omega) = e^{\diamond X(x;\omega)}$, the *Wick exponential* of a random field X as the permeability; in this case measurement data suggests to take X as the log-correlated Gaussian field.

Motivation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- The *stochastic pressure equation*: $U \subset \mathbb{R}^n$ a bounded, smooth domain, consider

$$-\nabla_x \cdot (p(x;\omega) \diamond \nabla_x u(x;\omega)) = 0, \qquad x \in U, \omega \in \Omega,$$

  with some (non-random) boundary data. Here $\diamond$ is the so-called *Wick product*; acts as a stochastic renormalization of sorts
- Related to the ST1 Deep Heat project in Espoo, models water seeping through a porous medium. The plot of the solution looks like a percolation diagram
- Modelling suggests to take $p(x;\omega) = e^{\diamond X(x;\omega)}$, the *Wick exponential* of a random field X as the permeability; in this case measurement data suggests to take X as the log-correlated Gaussian field.

Motivation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- The *stochastic pressure equation*: $U \subset \mathbb{R}^n$ a bounded, smooth domain, consider

$$-\nabla_x \cdot (p(x;\omega) \diamond \nabla_x u(x;\omega)) = 0, \qquad x \in U, \omega \in \Omega,$$

  with some (non-random) boundary data. Here $\diamond$ is the so-called *Wick product*; acts as a stochastic renormalization of sorts
- Related to the ST1 Deep Heat project in Espoo, models water seeping through a porous medium. The plot of the solution looks like a percolation diagram
- Modelling suggests to take $p(x;\omega) = e^{\diamond X(x;\omega)}$, the *Wick exponential* of a random field X as the permeability; in this case measurement data suggests to take X as the log-correlated Gaussian field.

Quick recap of probability theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A *random vector* X is a measurable mapping $(\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. When $n = 1, X$ is called a random variable.
- We define the *characteristic function* of X (the Fourier transform of $Q_X = \mathbb{P} \circ X^{-1}$)

$$\hat{f}(t) = \int_\Omega e^{it \cdot X(\omega)}\, dQ_X(\omega), \qquad t \in \mathbb{R}^n,$$

  where the probability measure $Q_X : \mathcal{B}(\mathbb{R}^n) \to [0,1]$ is called the *distribution* of X.
- Let T be a set and $(E, \mathcal{E})$ a measurable space. A *random field* indexed by T is a collection $(X_t : t \in T)$ of random vectors $X_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$.
- Typically we take $T = \mathbb{R}^n$. When $n = 1$, X is referred to as a *stochastic process* rather than a field.
- Fix $\omega \in \Omega$. The map $t \mapsto X_t(\omega), t \in T$ is called the *(sample) path* of the random field.

Quick recap of probability theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A *random vector* X is a measurable mapping $(\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. When $n = 1, X$ is called a random variable.
- We define the *characteristic function* of X (the Fourier transform of $Q_X = \mathbb{P} \circ X^{-1}$)

$$\hat{f}(t) = \int_\Omega e^{it \cdot X(\omega)}\, dQ_X(\omega), \qquad t \in \mathbb{R}^n,$$

  where the probability measure $Q_X : \mathcal{B}(\mathbb{R}^n) \to [0, 1]$ is called the *distribution* of X.
- Let T be a set and $(E, \mathcal{E})$ a measurable space. A *random field* indexed by T is a collection $(X_t : t \in T)$ of random vectors $X_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$.
- Typically we take $T = \mathbb{R}^n$. When $n = 1$, X is referred to as a *stochastic process* rather than a field.
- Fix $\omega \in \Omega$. The map $t \mapsto X_t(\omega), t \in T$ is called the *(sample) path* of the random field.

Quick recap of probability theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A *random vector* X is a measurable mapping $(\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. When $n = 1, X$ is called a random variable.
- We define the *characteristic function* of X (the Fourier transform of $Q_X = \mathbb{P} \circ X^{-1}$)

$$\hat{f}(t) = \int_\Omega e^{it \cdot X(\omega)}\, dQ_X(\omega), \qquad t \in \mathbb{R}^n,$$

  where the probability measure $Q_X \colon \mathcal{B}(\mathbb{R}^n) \to [0, 1]$ is called the *distribution* of X.
- Let T be a set and $(E, \mathcal{E})$ a measurable space. A *random field* indexed by T is a collection $(X_t : t \in T)$ of random vectors $X_t \colon (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$.
- Typically we take $T = \mathbb{R}^n$. When $n = 1$, X is referred to as a *stochastic process* rather than a field.
- Fix $\omega \in \Omega$. The map $t \mapsto X_t(\omega), t \in T$ is called the *(sample) path* of the random field.

Quick recap of probability theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A *random vector* X is a measurable mapping $(\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. When $n = 1, X$ is called a random variable.
- We define the *characteristic function* of X (the Fourier transform of $Q_X = \mathbb{P} \circ X^{-1}$)

$$\hat{f}(t) = \int_\Omega e^{it \cdot X(\omega)} \, dQ_X(\omega), \qquad t \in \mathbb{R}^n,$$

  where the probability measure $Q_X \colon \mathcal{B}(\mathbb{R}^n) \to [0, 1]$ is called the *distribution* of X.
- Let T be a set and $(E, \mathcal{E})$ a measurable space. A *random field* indexed by T is a collection $(X_t : t \in T)$ of random vectors $X_t \colon (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$.
- Typically we take $T = \mathbb{R}^n$. When $n = 1$, X is referred to as a *stochastic process* rather than a field.
- Fix $\omega \in \Omega$. The map $t \mapsto X_t(\omega), t \in T$ is called the *(sample) path* of the random field.

Quick recap of probability theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A *random vector* X is a measurable mapping $(\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. When $n = 1, X$ is called a random variable.
- We define the *characteristic function* of X (the Fourier transform of $Q_X = \mathbb{P} \circ X^{-1}$)

$$\hat{f}(t) = \int_\Omega e^{it \cdot X(\omega)} \, dQ_X(\omega), \qquad t \in \mathbb{R}^n,$$

  where the probability measure $Q_X \colon \mathcal{B}(\mathbb{R}^n) \to [0, 1]$ is called the *distribution* of X.
- Let T be a set and $(E, \mathcal{E})$ a measurable space. A *random field* indexed by T is a collection $(X_t : t \in T)$ of random vectors $X_t \colon (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$.
- Typically we take $T = \mathbb{R}^n$. When $n = 1$, X is referred to as a *stochastic process* rather than a field.
- Fix $\omega \in \Omega$. The map $t \mapsto X_t(\omega), t \in T$ is called the *(sample) path* of the random field.

Quick recap of probability theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A *random vector* X is a measurable mapping $(\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. When $n = 1, X$ is called a random variable.
- We define the *characteristic function* of X (the Fourier transform of $Q_X = \mathbb{P} \circ X^{-1}$)

$$\hat{f}(t) = \int_\Omega e^{it\cdot X(\omega)}\, dQ_X(\omega), \qquad t \in \mathbb{R}^n,$$

  where the probability measure $Q_X : \mathcal{B}(\mathbb{R}^n) \to [0,1]$ is called the *distribution* of X.
- Let T be a set and $(E, \mathcal{E})$ a measurable space. A *random field* indexed by T is a collection $(X_t : t \in T)$ of random vectors $X_t : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{E})$.
- Typically we take $T = \mathbb{R}^n$. When $n = 1$, X is referred to as a *stochastic process* rather than a field.
- Fix $\omega \in \Omega$. The map $t \mapsto X_t(\omega), t \in T$ is called the *(sample) path* of the random field.

Gaussian random variables

Why Gaussians?

- Normal distribution shows up often in applications, even when not assuming a priori Gaussianity, due to Central Limit Theorem -type results
- In many ways the "easiest" case (cf. linear vs non-linear DE): In particular Gaussians are stable under linear transformation and limits
- Many of the Gaussian techniques can be adapted with some modification for more difficult distributions
- One can obtain many other distributions from transformations of Gaussians

Gaussian random variables

Why Gaussians?

- Normal distribution shows up often in applications, even when not assuming a priori Gaussianity, due to Central Limit Theorem -type results
- In many ways the "easiest" case (cf. linear vs non-linear DE): In particular Gaussians are stable under linear transformation and limits
- Many of the Gaussian techniques can be adapted with some modification for more difficult distributions
- One can obtain many other distributions from transformations of Gaussians

Gaussian random variables

Why Gaussians?

- Normal distribution shows up often in applications, even when not assuming a priori Gaussianity, due to Central Limit Theorem -type results
- In many ways the "easiest" case (cf. linear vs non-linear DE): In particular Gaussians are stable under linear transformation and limits
- Many of the Gaussian techniques can be adapted with some modification for more difficult distributions
- One can obtain many other distributions from transformations of Gaussians

Gaussian random variables

Why Gaussians?

- Normal distribution shows up often in applications, even when not assuming a priori Gaussianity, due to Central Limit Theorem -type results
- In many ways the "easiest" case (cf. linear vs non-linear DE): In particular Gaussians are stable under linear transformation and limits
- Many of the Gaussian techniques can be adapted with some modification for more difficult distributions
- One can obtain many other distributions from transformations of Gaussians

Gaussian random variables

Why Gaussians?

- Normal distribution shows up often in applications, even when not assuming a priori Gaussianity, due to Central Limit Theorem -type results
- In many ways the "easiest" case (cf. linear vs non-linear DE): In particular Gaussians are stable under linear transformation and limits
- Many of the Gaussian techniques can be adapted with some modification for more difficult distributions
- One can obtain many other distributions from transformations of Gaussians

Gaussian random variables: the finite-dimensional case

Definition

A *Gaussian random variable* X is a real-valued random variable with characteristic function $\hat{f}(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$ for some $\mu \in \mathbb{R}, \sigma^2 > 0$. The parameter μ is called the mean, σ^2 is called the variance of X.

- X is said to be *centred* if $\mu = 0$; X is *standard Gaussian* if $\mu = 0, \sigma^2 = 1$.
- A random vector $X = (X_1, ..., X_n)$ with values in $\mathbb{R}^n$ is said to be *Gaussian* if $X \cdot t \in \mathbb{R}$ is a Gaussian random variable for all $t \in \mathbb{R}^n$.

Gaussian random variables: the finite-dimensional case

Definition

A *Gaussian random variable* X is a real-valued random variable with characteristic function $\hat{f}(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$ for some $\mu \in \mathbb{R}, \sigma^2 > 0$. The parameter μ is called the mean, σ^2 is called the variance of X.

- X is said to be *centred* if $\mu = 0$; X is *standard Gaussian* if $\mu = 0, \sigma^2 = 1$.
- A random vector $X = (X_1, ..., X_n)$ with values in $\mathbb{R}^n$ is said to be *Gaussian* if $X \cdot t \in \mathbb{R}$ is a Gaussian random variable for all $t \in \mathbb{R}^n$.

Gaussian random variables: the finite-dimensional case

Definition

A *Gaussian random variable* X is a real-valued random variable with characteristic function $\hat{f}(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$ for some $\mu \in \mathbb{R}, \sigma^2 > 0$. The parameter μ is called the mean, σ^2 is called the variance of X.

- X is said to be *centred* if $\mu = 0$; X is *standard Gaussian* if $\mu = 0, \sigma^2 = 1$.
- A random vector $X = (X_1, ..., X_n)$ with values in $\mathbb{R}^n$ is said to be *Gaussian* if $X \cdot t \in \mathbb{R}$ is a Gaussian random variable for all $t \in \mathbb{R}^n$.

Gaussians: the infinite-dimensional (Hilbert) case

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$.

- A random variable X with values in H is said to be *Gaussian* if the real-valued random variable $(h, X)_H$ is Gaussian for every $h \in H$.
- Characterisation of Gaussians: if X is a centred random variable with values in H, the following can be shown to be equivalent:
  1. X is Gaussian;
  2. There exists a positive, symmetric and continuous linear operator $Q \colon H \to H$ such that the Fourier transform of X is given by

$$\mathbb{E}[\exp(i(X,h)_H)] = \exp\left(-\frac{1}{2}(Qh,h)_H\right), \qquad \forall h \in H.$$

  Moreover, the operator Q, called the covariance operator, is uniquely determined by (2).
- Moments of Gaussians (Fernique thm) Let X be a Hilbert-space valued Gaussian. There exists a constant $\beta > 0$ such that $\mathbb{E}[\exp(\beta\|X\|_H^2)] < +\infty$.
- Independent Gaussians form a linear space.
- Fact: $L^2(\Omega)$-limit of Gaussians is Gaussian; moreover, for Gaussian random variables many forms of convergence are actually equivalent.

Gaussians: the infinite-dimensional (Hilbert) case

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$.

- A random variable X with values in H is said to be *Gaussian* if the real-valued random variable $(h, X)_H$ is Gaussian for every $h \in H$.
- Characterisation of Gaussians: if X is a centred random variable with values in H, the following can be shown to be equivalent:
  1. X is Gaussian;
  2. There exists a positive, symmetric and continuous linear operator $Q\colon H \to H$ such that the Fourier transform of X is given by

$$\mathbb{E}[\exp(i(X,h)_H)] = \exp\left(-\frac{1}{2}(Qh,h)_H\right), \qquad \forall h \in H.$$

  Moreover, the operator Q, called the covariance operator, is uniquely determined by (2).
- Moments of Gaussians (Fernique thm) Let X be a Hilbert-space valued Gaussian. There exists a constant $\beta > 0$ such that $\mathbb{E}[\exp(\beta\|X\|_H^2)] < +\infty$.
- Independent Gaussians form a linear space.
- Fact: $L^2(\Omega)$-limit of Gaussians is Gaussian; moreover, for Gaussian random variables many forms of convergence are actually equivalent.

Gaussians: the infinite-dimensional (Hilbert) case

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$.

- A random variable X with values in H is said to be *Gaussian* if the real-valued random variable $(h, X)_H$ is Gaussian for every $h \in H$.
- Characterisation of Gaussians: if X is a centred random variable with values in H, the following can be shown to be equivalent:
  1. X is Gaussian;
  2. There exists a positive, symmetric and continuous linear operator $Q\colon H \to H$ such that the Fourier transform of X is given by

$$\mathbb{E}[\exp(i(X,h)_H)] = \exp\left(-\frac{1}{2}(Qh,h)_H\right), \qquad \forall h \in H.$$

  Moreover, the operator Q, called the covariance operator, is uniquely determined by (2).
- Moments of Gaussians (Fernique thm) Let X be a Hilbert-space valued Gaussian. There exists a constant $\beta > 0$ such that $\mathbb{E}[\exp(\beta\|X\|_H^2)] < +\infty$.
- Independent Gaussians form a linear space.
- Fact: $L^2(\Omega)$-limit of Gaussians is Gaussian; moreover, for Gaussian random variables many forms of convergence are actually equivalent.

Gaussians: the infinite-dimensional (Hilbert) case

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$.

- A random variable X with values in H is said to be *Gaussian* if the real-valued random variable $(h, X)_H$ is Gaussian for every $h \in H$.
- Characterisation of Gaussians: if X is a centred random variable with values in H, the following can be shown to be equivalent:
  1. X is Gaussian;
  2. There exists a positive, symmetric and continuous linear operator $Q\colon H \to H$ such that the Fourier transform of X is given by

$$\mathbb{E}[\exp(i(X,h)_H)] = \exp\left(-\frac{1}{2}(Qh,h)_H\right), \qquad \forall h \in H.$$

  Moreover, the operator Q, called the covariance operator, is uniquely determined by (2).
- Moments of Gaussians (Fernique thm) Let X be a Hilbert-space valued Gaussian. There exists a constant $\beta > 0$ such that $\mathbb{E}[\exp(\beta\|X\|_H^2)] < +\infty$.
- Independent Gaussians form a linear space.
- Fact: $L^2(\Omega)$-limit of Gaussians is Gaussian; moreover, for Gaussian random variables many forms of convergence are actually equivalent.

Gaussians: the infinite-dimensional (Hilbert) case

Let H be a real Hilbert space with scalar product $(\cdot,\cdot)_H$.

- A random variable X with values in H is said to be *Gaussian* if the real-valued random variable $(h, X)_H$ is Gaussian for every $h \in H$.
- Characterisation of Gaussians: if X is a centred random variable with values in H, the following can be shown to be equivalent:
  1. X is Gaussian;
  2. There exists a positive, symmetric and continuous linear operator $Q\colon H \to H$ such that the Fourier transform of X is given by

$$\mathbb{E}[\exp(i(X,h)_H)] = \exp\left(-\frac{1}{2}(Qh,h)_H\right), \qquad \forall h \in H.$$

  Moreover, the operator Q, called the covariance operator, is uniquely determined by (2).
- Moments of Gaussians (Fernique thm) Let X be a Hilbert-space valued Gaussian. There exists a constant $\beta > 0$ such that $\mathbb{E}[\exp(\beta\|X\|_H^2)] < +\infty$.
- Independent Gaussians form a linear space.
- Fact: $L^2(\Omega)$-limit of Gaussians is Gaussian; moreover, for Gaussian random variables many forms of convergence are actually equivalent.

Examples of Gaussian processes (I)

- *Karhunen-Loève expansion.* Let H be a separable Hilbert space. Let $(e_j) \subset H$ be an orthonormal basis, (ξ_j) a sequence of real-valued independent, standard Gaussian random variables and $(\sigma_j) \subset \mathbb{R}_+$ satisfying $\sum_{j=1}^{\infty} \sigma_j^2 < +\infty$. Then any Gaussian $X \in H$ can be written as $X = \sum_{j=1}^{\infty} \sigma_j \xi_j e_j$, where the sum converges almost surely with respect to $\| \cdot \|_H$.
- *Brownian motion.* Let (ξ_i) be independent standard Gaussian random variables. For each n, define a random step process $W_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k, t \in [0,1]$. On the limit as $n \to \infty$ we obtain a process called the Brownian motion.
- Brownian motion (W_t) satisfies $\mathbb{E}[W_t] = 0, \mathrm{Var}(W_t) = t, \mathrm{Cov}(W_t, W_s) = t \wedge s$, for all $s, t \in [0,1]$.

Examples of Gaussian processes (I)

- *Karhunen-Loève expansion.* Let H be a separable Hilbert space. Let $(e_j) \subset H$ be an orthonormal basis, (ξ_j) a sequence of real-valued independent, standard Gaussian random variables and $(\sigma_j) \subset \mathbb{R}_+$ satisfying $\sum_{j=1}^{\infty} \sigma_j^2 < +\infty$. Then any Gaussian $X \in H$ can be written as $X = \sum_{j=1}^{\infty} \sigma_j \xi_j e_j$, where the sum converges almost surely with respect to $\|\cdot\|_H$.
- *Brownian motion.* Let (ξ_i) be independent standard Gaussian random variables. For each n, define a random step process $W_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k, t \in [0,1]$. On the limit as $n \to \infty$ we obtain a process called the Brownian motion.
- Brownian motion (W_t) satisfies $\mathbb{E}[W_t] = 0, \mathrm{Var}(W_t) = t, \mathrm{Cov}(W_t, W_s) = t \wedge s$, for all $s, t \in [0,1]$.

Examples of Gaussian processes (I)

- *Karhunen-Loève expansion.* Let H be a separable Hilbert space. Let $(e_j) \subset H$ be an orthonormal basis, (ξ_j) a sequence of real-valued independent, standard Gaussian random variables and $(\sigma_j) \subset \mathbb{R}_+$ satisfying $\sum_{j=1}^{\infty} \sigma_j^2 < +\infty$. Then any Gaussian $X \in H$ can be written as $X = \sum_{j=1}^{\infty} \sigma_j \xi_j e_j$, where the sum converges almost surely with respect to $\|\cdot\|_H$.
- *Brownian motion.* Let (ξ_i) be independent standard Gaussian random variables. For each n, define a random step process $W_n(t) := \frac{1}{\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k, t \in [0, 1]$. On the limit as $n \to \infty$ we obtain a process called the Brownian motion.
- Brownian motion (W_t) satisfies $\mathbb{E}[W_t] = 0, \mathrm{Var}(W_t) = t, \mathrm{Cov}(W_t, W_s) = t \wedge s$, for all $s, t \in [0, 1]$.

Examples of Gaussian processes (II)

- Fractional Brownian motion (FBm). A continuous time Gaussian process B^H on $[0, 1]$ which satisfies $\mathbb{E}[B^H(0)] = 0$, $\mathbb{E}[B^H(t)] = 0, \forall t \in [0, 1]$, and

$$\mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t - s|^{2H}\right) \quad \forall t \in [0, 1],$$

  where $H \in (0, 1)$ is called the *Hurst index* (for $H = 1/2$ one obtains the Brownian motion).
- Regularity of FBm: sample paths of B^H (i.e. functions $t \mapsto B^H(t, \omega)$) are nowhere differentiable. However, the sample path of B^H is Hölder with index $H - \varepsilon$, for every $\varepsilon > 0$.
- One may also consider several dimensional variants of BM and FBm taking their values in $\mathbb{R}^n$, for instance by requiring that each of the projections $B_i^H, i = 1, ..., n$, are one-dimensional Fractional Brownian motions.

Examples of Gaussian processes (II)

- Fractional Brownian motion (FBm). A continuous time Gaussian process B^H on $[0,1]$ which satisfies $\mathbb{E}[B^H(0)] = 0$, $\mathbb{E}[B^H(t)] = 0, \forall t \in [0,1]$, and

$$\mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \quad \forall t \in [0,1],$$

  where $H \in (0,1)$ is called the *Hurst index* (for $H = 1/2$ one obtains the Brownian motion).
- Regularity of FBm: sample paths of B^H (i.e. functions $t \mapsto B^H(t,\omega)$) are nowhere differentiable. However, the sample path of B^H is Hölder with index $H - \varepsilon$, for every $\varepsilon > 0$.
- One may also consider several dimensional variants of BM and FBm taking their values in $\mathbb{R}^n$, for instance by requiring that each of the projections $B_i^H, i = 1, ..., n$, are one-dimensional Fractional Brownian motions.

Examples of Gaussian processes (II)

- Fractional Brownian motion (FBm). A continuous time Gaussian process B^H on $[0,1]$ which satisfies $\mathbb{E}[B^H(0)] = 0$, $\mathbb{E}[B^H(t)] = 0, \forall t \in [0,1]$, and

$$\mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \quad \forall t \in [0,1],$$

  where $H \in (0,1)$ is called the *Hurst index* (for $H = 1/2$ one obtains the Brownian motion).
- Regularity of FBm: sample paths of B^H (i.e. functions $t \mapsto B^H(t,\omega)$) are nowhere differentiable. However, the sample path of B^H is Hölder with index $H - \varepsilon$, for every $\varepsilon > 0$.
- One may also consider several dimensional variants of BM and FBm taking their values in $\mathbb{R}^n$, for instance by requiring that each of the projections $B_i^H, i = 1, ..., n$, are one-dimensional Fractional Brownian motions.

Plots of FBm sample paths with varying Hurst index

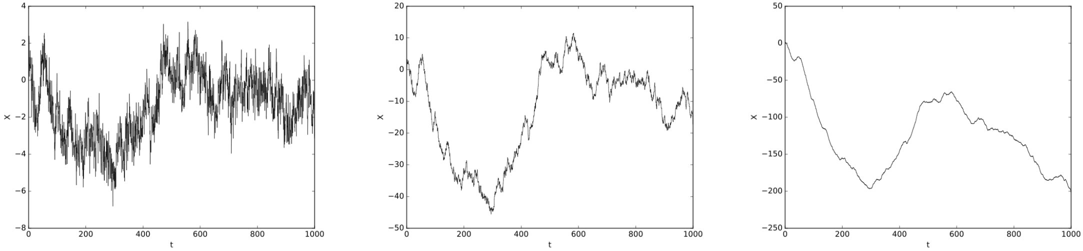

Figure: Simulations of sample paths of a single realisation of FBm ("with fixed $\omega \in \Omega$") with Hurst parameter $H = 0.15, H = 0.55, H = 0.95$

Gaussian Hilbert space. Construction of white noise (I)

- Recall the *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$, the space of (smooth) functions on $\mathbb{R}^n$ whose derivatives are rapidly decreasing, and its dual $\mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*.
- We say that a complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ if $\Phi(\phi) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(x,\phi)}\, d\nu(x), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
- Bochner-Minlos Theorem for $\Omega = \mathcal{S}'(\mathbb{R}^n)$: *A complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a (unique) probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ iff $\Phi(0) = 1$, Φ is continuous, and Φ is positive definite.*
- Define $\Phi_0(\phi) := \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. By Bochner-Minlos theorem, there exists a unique probability measure μ on $\mathcal{S}'$ having Φ_0 as its characteristic function; we call μ the *white noise measure*.
- Since we have $\int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(W,\phi)}\, d\mu(W) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$, and characteristic functions determine distributions, the random variable (W,ϕ) is a centred Gaussian with variance equal to $\|\phi\|^2_{L^2(\mathbb{R}^n)}$, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Gaussian Hilbert space. Construction of white noise (I)

- Recall the *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$, the space of (smooth) functions on $\mathbb{R}^n$ whose derivatives are rapidly decreasing, and its dual $\mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*.
- We say that a complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ if $\Phi(\phi) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(x,\phi)} \, d\nu(x), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
- Bochner-Minlos Theorem for $\Omega = \mathcal{S}'(\mathbb{R}^n)$: *A complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a (unique) probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ iff $\Phi(0) = 1$, Φ is continuous, and Φ is positive definite.*
- Define $\Phi_0(\phi) := \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. By Bochner-Minlos theorem, there exists a unique probability measure μ on $\mathcal{S}'$ having Φ_0 as its characteristic function; we call μ the *white noise measure*.
- Since we have $\int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(W,\phi)} \, d\mu(W) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$, and characteristic functions determine distributions, the random variable (W,ϕ) is a centred Gaussian with variance equal to $\|\phi\|^2_{L^2(\mathbb{R}^n)}$, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Gaussian Hilbert space. Construction of white noise (I)

- Recall the *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$, the space of (smooth) functions on $\mathbb{R}^n$ whose derivatives are rapidly decreasing, and its dual $\mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*.
- We say that a complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ if $\Phi(\phi) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(x,\phi)}\, d\nu(x), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
- Bochner-Minlos Theorem for $\Omega = \mathcal{S}'(\mathbb{R}^n)$: *A complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a (unique) probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ iff $\Phi(0) = 1$, Φ is continuous, and Φ is positive definite.*
- Define $\Phi_0(\phi) := \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. By Bochner-Minlos theorem, there exists a unique probability measure μ on $\mathcal{S}'$ having Φ_0 as its characteristic function; we call μ the *white noise measure*.
- Since we have $\int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(W,\phi)}\, d\mu(W) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$, and characteristic functions determine distributions, the random variable (W,ϕ) is a centred Gaussian with variance equal to $\|\phi\|^2_{L^2(\mathbb{R}^n)}$, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Gaussian Hilbert space. Construction of white noise (I)

- Recall the *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$, the space of (smooth) functions on $\mathbb{R}^n$ whose derivatives are rapidly decreasing, and its dual $\mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*.
- We say that a complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ if $\Phi(\phi) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(x,\phi)}\, d\nu(x), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
- Bochner-Minlos Theorem for $\Omega = \mathcal{S}'(\mathbb{R}^n)$: *A complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a (unique) probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ iff $\Phi(0) = 1$, Φ is continuous, and Φ is positive definite.*
- Define $\Phi_0(\phi) := \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. By Bochner-Minlos theorem, there exists a unique probability measure μ on $\mathcal{S}'$ having Φ_0 as its characteristic function; we call μ the *white noise measure*.
- Since we have $\int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(W,\phi)}\, d\mu(W) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$, and characteristic functions determine distributions, the random variable (W,ϕ) is a centred Gaussian with variance equal to $\|\phi\|^2_{L^2(\mathbb{R}^n)}$, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Gaussian Hilbert space. Construction of white noise (I)

- Recall the *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$, the space of (smooth) functions on $\mathbb{R}^n$ whose derivatives are rapidly decreasing, and its dual $\mathcal{S}'(\mathbb{R}^n)$, the space of *tempered distributions*.
- We say that a complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ if $\Phi(\phi) = \int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(x,\phi)}\, d\nu(x), \quad \forall \phi \in \mathcal{S}(\mathbb{R}^n)$.
- Bochner-Minlos Theorem for $\Omega = \mathcal{S}'(\mathbb{R}^n)$: *A complex-valued function Φ on $\mathcal{S}(\mathbb{R}^n)$ is the characteristic function of a (unique) probability measure ν on $\mathcal{S}'(\mathbb{R}^n)$ iff $\Phi(0) = 1$, Φ is continuous, and Φ is positive definite.*
- Define $\Phi_0(\phi) := \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. By Bochner-Minlos theorem, there exists a unique probability measure μ on $\mathcal{S}'$ having Φ_0 as its characteristic function; we call μ the *white noise measure*.
- Since we have $\int_{\mathcal{S}'(\mathbb{R}^n)} e^{i(W,\phi)}\, d\mu(W) = \exp\left(-\frac{1}{2}\|\phi\|^2_{L^2(\mathbb{R}^n)}\right)$, and characteristic functions determine distributions, the random variable (W,ϕ) is a centred Gaussian with variance equal to $\|\phi\|^2_{L^2(\mathbb{R}^n)}$, for any $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Gaussian Hilbert space. Construction of white noise (II)

Definition

A *Gaussian Hilbert space* is a collection of Gaussian random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ inner product and closed with respect to the corresponding norm.

- To define a Gaussian Hilbert space $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$, consider the map from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\Omega)$ given by $\phi \mapsto (W, \phi)$. Since $\mathbb{E}[(W, \phi)^2] = \|\phi\|^2_{L^2(\mathbb{R}^n)}$, this map is an isometry.
- As $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, and $L^2(\Omega)$ is complete, we extend this operator to a one from $L^2(\mathbb{R}^n)$ to $L^2(\Omega)$ by defining $(W, f) := \lim_{n\to\infty}(W, \phi_n)$, where $\phi_n \in \mathcal{S}(\mathbb{R}^n)$ and $\phi_n \to f$ in $L^2(\mathbb{R}^n)$.
- By dominated convergence $\mathbb{E}[e^{i(W,\phi_n)}] \to \mathbb{E}[e^{i(W,f)}]$; Hence the random variable (W, f) is centred Gaussian with variance equal to $\|f\|^2_{L^2(\mathbb{R}^n)}$.
- We call $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$ a *white noise Gaussian Hilbert space.*

Gaussian Hilbert space. Construction of white noise (II)

Definition

A *Gaussian Hilbert space* is a collection of Gaussian random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ inner product and closed with respect to the corresponding norm.

- To define a Gaussian Hilbert space $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$, consider the map from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\Omega)$ given by $\phi \mapsto (W, \phi)$. Since $\mathbb{E}[(W, \phi)^2] = \|\phi\|^2_{L^2(\mathbb{R}^n)}$, this map is an isometry.
- As $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, and $L^2(\Omega)$ is complete, we extend this operator to a one from $L^2(\mathbb{R}^n)$ to $L^2(\Omega)$ by defining $(W, f) := \lim_{n\to\infty}(W, \phi_n)$, where $\phi_n \in \mathcal{S}(\mathbb{R}^n)$ and $\phi_n \to f$ in $L^2(\mathbb{R}^n)$.
- By dominated convergence $\mathbb{E}[e^{i(W,\phi_n)}] \to \mathbb{E}[e^{i(W,f)}]$; Hence the random variable (W, f) is centred Gaussian with variance equal to $\|f\|^2_{L^2(\mathbb{R}^n)}$.
- We call $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$ a *white noise Gaussian Hilbert space.*

Gaussian Hilbert space. Construction of white noise (II)

Definition

A *Gaussian Hilbert space* is a collection of Gaussian random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ inner product and closed with respect to the corresponding norm.

- To define a Gaussian Hilbert space $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$, consider the map from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\Omega)$ given by $\phi \mapsto (W, \phi)$. Since $\mathbb{E}[(W, \phi)^2] = \|\phi\|^2_{L^2(\mathbb{R}^n)}$, this map is an isometry.
- As $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, and $L^2(\Omega)$ is complete, we extend this operator to a one from $L^2(\mathbb{R}^n)$ to $L^2(\Omega)$ by defining $(W, f) := \lim_{n\to\infty}(W, \phi_n)$, where $\phi_n \in \mathcal{S}(\mathbb{R}^n)$ and $\phi_n \to f$ in $L^2(\mathbb{R}^n)$.
- By dominated convergence $\mathbb{E}[e^{i(W,\phi_n)}] \to \mathbb{E}[e^{i(W,f)}]$; Hence the random variable (W, f) is centred Gaussian with variance equal to $\|f\|^2_{L^2(\mathbb{R}^n)}$.
- We call $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$ a *white noise Gaussian Hilbert space.*

Gaussian Hilbert space. Construction of white noise (II)

Definition

A *Gaussian Hilbert space* is a collection of Gaussian random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ inner product and closed with respect to the corresponding norm.

- To define a Gaussian Hilbert space $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$, consider the map from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\Omega)$ given by $\phi \mapsto (W, \phi)$. Since $\mathbb{E}[(W, \phi)^2] = \|\phi\|^2_{L^2(\mathbb{R}^n)}$, this map is an isometry.
- As $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, and $L^2(\Omega)$ is complete, we extend this operator to a one from $L^2(\mathbb{R}^n)$ to $L^2(\Omega)$ by defining $(W, f) := \lim_{n\to\infty}(W, \phi_n)$, where $\phi_n \in \mathcal{S}(\mathbb{R}^n)$ and $\phi_n \to f$ in $L^2(\mathbb{R}^n)$.
- By dominated convergence $\mathbb{E}[e^{i(W,\phi_n)}] \to \mathbb{E}[e^{i(W,f)}]$; Hence the random variable (W, f) is centred Gaussian with variance equal to $\|f\|^2_{L^2(\mathbb{R}^n)}$.
- We call $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$ a *white noise Gaussian Hilbert space.*

Gaussian Hilbert space. Construction of white noise (II)

Definition

A *Gaussian Hilbert space* is a collection of Gaussian random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the $L^2(\Omega, \mathcal{F}, \mathbb{P})$ inner product and closed with respect to the corresponding norm.

- To define a Gaussian Hilbert space $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$, consider the map from $\mathcal{S}(\mathbb{R}^n)$ to $L^2(\Omega)$ given by $\phi \mapsto (W, \phi)$. Since $\mathbb{E}[(W, \phi)^2] = \|\phi\|^2_{L^2(\mathbb{R}^n)}$, this map is an isometry.
- As $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, and $L^2(\Omega)$ is complete, we extend this operator to a one from $L^2(\mathbb{R}^n)$ to $L^2(\Omega)$ by defining $(W, f) := \lim_{n\to\infty}(W, \phi_n)$, where $\phi_n \in \mathcal{S}(\mathbb{R}^n)$ and $\phi_n \to f$ in $L^2(\mathbb{R}^n)$.
- By dominated convergence $\mathbb{E}[e^{i(W,\phi_n)}] \to \mathbb{E}[e^{i(W,f)}]$; Hence the random variable (W, f) is centred Gaussian with variance equal to $\|f\|^2_{L^2(\mathbb{R}^n)}$.
- We call $\{(W, f) : f \in L^2(\mathbb{R}^n)\}$ a *white noise Gaussian Hilbert space*.

Gaussian Hilbert space. Construction of white noise (III)

- For $f, g \in L^2(\mathbb{R}^n)$, we may apply this to $(W, f+g)$ to see that $\mathrm{Cov}[(W,f),(W,g)] = (f,g)_{L^2(\mathbb{R}^n)}$; i.e. if f, g are orthogonal in $L^2(\mathbb{R}^n)$, the (Gaussian) random variables $(W,f), (W,g)$ are independent.
- We formally rewrite the above as $\mathrm{Cov}[(W,f),(W,g)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \delta(x-y) f(x) g(y)\, dx\, dy$, and say that W has the *covariance kernel* $\delta(x-y)$, where δ denotes the Dirac measure.
- One may also define stochastic processes on the white noise space. Example: let $n = 1$, setting $B_t(W) := (W, \mathbb{1}_{(0,t)})$, where $t \mapsto \mathbb{1}_{(a,b)}(t) \in L^2(\mathbb{R})$ is the indicator function of (a,b), can be shown to yield a Brownian motion process.
- Construction of other fractional Gaussian Hilbert spaces: find a suitable characteristic functional, some Hilbert space and a dense subspace (here we used L^2 and $\mathcal{S}$); use Bochner-Minlos thm; extend isometrically.

Gaussian Hilbert space. Construction of white noise (III)

- For $f, g \in L^2(\mathbb{R}^n)$, we may apply this to $(W, f+g)$ to see that $\mathrm{Cov}[(W,f),(W,g)] = (f,g)_{L^2(\mathbb{R}^n)}$; i.e. if f, g are orthogonal in $L^2(\mathbb{R}^n)$, the (Gaussian) random variables $(W,f),(W,g)$ are independent.
- We formally rewrite the above as $\mathrm{Cov}[(W,f),(W,g)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \delta(x-y) f(x) g(y)\, dx\, dy$, and say that W has the *covariance kernel* $\delta(x-y)$, where δ denotes the Dirac measure.
- One may also define stochastic processes on the white noise space. Example: let $n=1$, setting $B_t(W) := (W, \mathbb{1}_{(0,t)})$, where $t \mapsto \mathbb{1}_{(a,b)}(t) \in L^2(\mathbb{R})$ is the indicator function of (a,b), can be shown to yield a Brownian motion process.
- Construction of other fractional Gaussian Hilbert spaces: find a suitable characteristic functional, some Hilbert space and a dense subspace (here we used L^2 and $\mathcal{S}$); use Bochner-Minlos thm; extend isometrically.

Gaussian Hilbert space. Construction of white noise (III)

- For $f, g \in L^2(\mathbb{R}^n)$, we may apply this to $(W, f+g)$ to see that $\mathrm{Cov}[(W,f),(W,g)] = (f,g)_{L^2(\mathbb{R}^n)}$; i.e. if f, g are orthogonal in $L^2(\mathbb{R}^n)$, the (Gaussian) random variables $(W,f),(W,g)$ are independent.
- We formally rewrite the above as $\mathrm{Cov}[(W,f),(W,g)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \delta(x-y) f(x) g(y)\, dx\, dy$, and say that W has the *covariance kernel* $\delta(x-y)$, where δ denotes the Dirac measure.
- One may also define stochastic processes on the white noise space. Example: let $n=1$, setting $B_t(W) := (W, \mathbb{1}_{(0,t)})$, where $t \mapsto \mathbb{1}_{(a,b)}(t) \in L^2(\mathbb{R})$ is the indicator function of (a,b), can be shown to yield a Brownian motion process.
- Construction of other fractional Gaussian Hilbert spaces: find a suitable characteristic functional, some Hilbert space and a dense subspace (here we used L^2 and $\mathcal{S}$); use Bochner-Minlos thm; extend isometrically.

Gaussian Hilbert space. Construction of white noise (III)

- For $f, g \in L^2(\mathbb{R}^n)$, we may apply this to $(W, f+g)$ to see that $\mathrm{Cov}[(W,f),(W,g)] = (f,g)_{L^2(\mathbb{R}^n)}$; i.e. if f, g are orthogonal in $L^2(\mathbb{R}^n)$, the (Gaussian) random variables $(W,f),(W,g)$ are independent.
- We formally rewrite the above as $\mathrm{Cov}[(W,f),(W,g)] = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \delta(x-y) f(x) g(y)\, dx\, dy$, and say that W has the *covariance kernel* $\delta(x-y)$, where δ denotes the Dirac measure.
- One may also define stochastic processes on the white noise space. Example: let $n = 1$, setting $B_t(W) := (W, \mathbb{1}_{(0,t)})$, where $t \mapsto \mathbb{1}_{(a,b)}(t) \in L^2(\mathbb{R})$ is the indicator function of (a,b), can be shown to yield a Brownian motion process.
- Construction of other fractional Gaussian Hilbert spaces: find a suitable characteristic functional, some Hilbert space and a dense subspace (here we used L^2 and $\mathcal{S}$); use Bochner-Minlos thm; extend isometrically.

Log-correlated Gaussian field (LGF)

Definition

The LGF is a centred real-valued Gaussian random tempered distribution h on $\mathbb{R}^n$, defined modulo a global additive constant, whose distribution is determined by the covariance

$$\mathrm{Cov}((h,\phi_1),(h,\phi_2)) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \left(\log\frac{1}{|y-z|} + g(y,z)\right)\phi_1(y)\phi_2(z)\,dy\,dz, \qquad \phi_1,\phi_2 \in \mathcal{S}_0(\mathbb{R}^n),$$

where $\mathcal{S}_0(\mathbb{R}^n) := \left\{f \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)\,dx = 0\right\}$ and g is a bounded function on $\mathbb{R}^n \times \mathbb{R}^n$. Here centred means that $\mathbb{E}[(h,\phi)] = 0$ for all $\phi \in \mathcal{S}_0(\mathbb{R}^n)$. A covariance kernel such as above is said to be of *log-type*.

- Equivalently one may define the LGF as $h = (-\Delta)^{-n/4}W$, where W is the white noise.
- When $n = 1$, the LGF can be shown to be morally the weak limit in $\mathcal{S}'(\mathbb{R})$ of FBm B^H as $H \to 0$. Recall: the paths of B^H belong to $C^{H-\varepsilon}(\mathbb{R})$ for all $\varepsilon > 0$; hence the LGF belongs to $C^{-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$.

Log-correlated Gaussian field (LGF)

Definition

The LGF is a centred real-valued Gaussian random tempered distribution h on $\mathbb{R}^n$, defined modulo a global additive constant, whose distribution is determined by the covariance

$$\mathrm{Cov}((h,\phi_1),(h,\phi_2)) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \left(\log\frac{1}{|y-z|} + g(y,z)\right)\phi_1(y)\phi_2(z)\,dy\,dz, \qquad \phi_1,\phi_2\in\mathcal{S}_0(\mathbb{R}^n),$$

where $\mathcal{S}_0(\mathbb{R}^n) := \left\{f\in\mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)\,dx = 0\right\}$ and g is a bounded function on $\mathbb{R}^n\times\mathbb{R}^n$. Here centred means that $\mathbb{E}[(h,\phi)] = 0$ for all $\phi\in\mathcal{S}_0(\mathbb{R}^n)$. A covariance kernel such as above is said to be of *log-type*.

- Equivalently one may define the LGF as $h = (-\Delta)^{-n/4}W$, where W is the white noise.
- When $n=1$, the LGF can be shown to be morally the weak limit in $\mathcal{S}'(\mathbb{R})$ of FBm B^H as $H\to 0$. Recall: the paths of B^H belong to $C^{H-\varepsilon}(\mathbb{R})$ for all $\varepsilon>0$; hence the LGF belongs to $C^{-\varepsilon}(\mathbb{R})$ for every $\varepsilon>0$.

Log-correlated Gaussian field (LGF)

Definition

The LGF is a centred real-valued Gaussian random tempered distribution h on $\mathbb{R}^n$, defined modulo a global additive constant, whose distribution is determined by the covariance

$$\mathrm{Cov}((h,\phi_1),(h,\phi_2)) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \left(\log\frac{1}{|y-z|} + g(y,z)\right)\phi_1(y)\phi_2(z)\,dy\,dz, \qquad \phi_1,\phi_2\in\mathcal{S}_0(\mathbb{R}^n),$$

where $\mathcal{S}_0(\mathbb{R}^n) := \left\{f\in\mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x)\,dx = 0\right\}$ and g is a bounded function on $\mathbb{R}^n\times\mathbb{R}^n$. Here centred means that $\mathbb{E}[(h,\phi)] = 0$ for all $\phi\in\mathcal{S}_0(\mathbb{R}^n)$. A covariance kernel such as above is said to be of *log-type*.

- Equivalently one may define the LGF as $h = (-\Delta)^{-n/4}W$, where W is the white noise.
- When $n=1$, the LGF can be shown to be morally the weak limit in $\mathcal{S}'(\mathbb{R})$ of FBm B^H as $H\to 0$. Recall: the paths of B^H belong to $C^{H-\varepsilon}(\mathbb{R})$ for all $\varepsilon>0$; hence the LGF belongs to $C^{-\varepsilon}(\mathbb{R})$ for every $\varepsilon>0$.

Gaussian multiplicative chaos measure (GMC)

- Roughly speaking, GMC is a theory which defines rigorously random measures

$$M_\gamma(dy) = e^{\gamma X(y)}\sigma(dy),$$

  where σ is a Radon measure on some metric space $(D, d), \gamma > 0$ is a parameter, and $X\colon D \to \mathbb{R}$ is a centred Gaussian field.
- Typically one constructs such measures from random fields X that are not defined as functions, so pointwise evaluations $X(y)$ do not make sense (LGF is an important example)
- The idea to construct a GMC measure is rather simple: define the measure as the limit as $\varepsilon \to 0$ of $C_\varepsilon e^{\gamma X_\varepsilon}\sigma(dx)$ where X_ε is a sequence converging to X, and C_ε is some normalisation sequence which ensures that the limit is non-trivial (i.e. we do not end up with the zero measure)

Gaussian multiplicative chaos measure (GMC)

- Roughly speaking, GMC is a theory which defines rigorously random measures

$$M_\gamma(dy) = e^{\gamma X(y)}\sigma(dy),$$

where σ is a Radon measure on some metric space $(D, d), \gamma > 0$ is a parameter, and $X\colon D \to \mathbb{R}$ is a centred Gaussian field.

- Typically one constructs such measures from random fields X that are not defined as functions, so pointwise evaluations $X(y)$ do not make sense (LGF is an important example)
- The idea to construct a GMC measure is rather simple: define the measure as the limit as $\varepsilon \to 0$ of $C_\varepsilon e^{\gamma X_\varepsilon}\sigma(dx)$ where X_ε is a sequence converging to X, and C_ε is some normalisation sequence which ensures that the limit is non-trivial (i.e. we do not end up with the zero measure)

Gaussian multiplicative chaos measure (GMC)

- Roughly speaking, GMC is a theory which defines rigorously random measures

$$M_\gamma(dy) = e^{\gamma X(y)}\sigma(dy),$$

where σ is a Radon measure on some metric space $(D, d), \gamma > 0$ is a parameter, and $X\colon D \to \mathbb{R}$ is a centred Gaussian field.

- Typically one constructs such measures from random fields X that are not defined as functions, so pointwise evaluations $X(y)$ do not make sense (LGF is an important example)
- The idea to construct a GMC measure is rather simple: define the measure as the limit as $\varepsilon \to 0$ of $C_\varepsilon e^{\gamma X_\varepsilon}\sigma(dx)$ where X_ε is a sequence converging to X, and C_ε is some normalisation sequence which ensures that the limit is non-trivial (i.e. we do not end up with the zero measure)

GMC measure. Construction.

- Let θ be a smooth mollifier. Set $X_\varepsilon := X * \theta_\varepsilon$, where X has a covariance kernel of log-type, and $\theta_\varepsilon := \frac{1}{\varepsilon^n}\theta\left(\frac{\cdot}{\varepsilon}\right)$. It can be shown that the random measures

$$M_{\varepsilon,\gamma}(dy) = e^{\gamma X_\varepsilon(y) - \frac{\gamma^2 \mathbb{E}[X_\varepsilon(y)^2]}{2}}\sigma(dy)$$

  converge in probability in the space of Radon measures towards a random measure M_γ.
- The random measure M_γ does not depend on the mollifier θ
- If $\sigma(dy) = f(y)\,dy$, with $f > 0$, the measure M_γ is different from 0 iff $\gamma < \sqrt{2n}$.
- The proof of this convergence is divided into two cases: $\gamma \in (0, \sqrt{n})$, the so-called L^2-range, which is easier, and then $\gamma \in [\sqrt{n}, \sqrt{2n})$, which uses more refined techniques.
- One could consider other, more general approximations of the field X rather than of the form $X * \theta_\varepsilon$ as well for the conclusion to hold

GMC measure. Construction.

- Let θ be a smooth mollifier. Set $X_\varepsilon := X * \theta_\varepsilon$, where X has a covariance kernel of log-type, and $\theta_\varepsilon := \frac{1}{\varepsilon^n}\theta\left(\frac{\cdot}{\varepsilon}\right)$. It can be shown that the random measures

$$M_{\varepsilon,\gamma}(dy) = e^{\gamma X_\varepsilon(y) - \frac{\gamma^2 \mathbb{E}[X_\varepsilon(y)^2]}{2}} \sigma(dy)$$

  converge in probability in the space of Radon measures towards a random measure M_γ.
- The random measure M_γ does not depend on the mollifier θ
- If $\sigma(dy) = f(y)\, dy$, with $f > 0$, the measure M_γ is different from 0 iff $\gamma < \sqrt{2n}$.
- The proof of this convergence is divided into two cases: $\gamma \in (0, \sqrt{n})$, the so-called L^2-range, which is easier, and then $\gamma \in [\sqrt{n}, \sqrt{2n})$, which uses more refined techniques.
- One could consider other, more general approximations of the field X rather than of the form $X * \theta_\varepsilon$ as well for the conclusion to hold

GMC measure. Construction.

- Let θ be a smooth mollifier. Set $X_\varepsilon := X * \theta_\varepsilon$, where X has a covariance kernel of log-type, and $\theta_\varepsilon := \frac{1}{\varepsilon^n}\theta\left(\frac{\cdot}{\varepsilon}\right)$. It can be shown that the random measures

$$M_{\varepsilon,\gamma}(dy) = e^{\gamma X_\varepsilon(y) - \frac{\gamma^2 \mathbb{E}[X_\varepsilon(y)^2]}{2}} \sigma(dy)$$

  converge in probability in the space of Radon measures towards a random measure M_γ.
- The random measure M_γ does not depend on the mollifier θ
- If $\sigma(dy) = f(y)\,dy$, with $f > 0$, the measure M_γ is different from 0 iff $\gamma < \sqrt{2n}$.
- The proof of this convergence is divided into two cases: $\gamma \in (0, \sqrt{n})$, the so-called L^2-range, which is easier, and then $\gamma \in [\sqrt{n}, \sqrt{2n})$, which uses more refined techniques.
- One could consider other, more general approximations of the field X rather than of the form $X * \theta_\varepsilon$ as well for the conclusion to hold

GMC measure. Construction.

- Let θ be a smooth mollifier. Set $X_\varepsilon := X * \theta_\varepsilon$, where X has a covariance kernel of log-type, and $\theta_\varepsilon := \frac{1}{\varepsilon^n}\theta\left(\frac{\cdot}{\varepsilon}\right)$. It can be shown that the random measures

$$M_{\varepsilon,\gamma}(dy) = e^{\gamma X_\varepsilon(y) - \frac{\gamma^2 \mathbb{E}[X_\varepsilon(y)^2]}{2}} \sigma(dy)$$

  converge in probability in the space of Radon measures towards a random measure M_γ.
- The random measure M_γ does not depend on the mollifier θ
- If $\sigma(dy) = f(y)\,dy$, with $f > 0$, the measure M_γ is different from 0 iff $\gamma < \sqrt{2n}$.
- The proof of this convergence is divided into two cases: $\gamma \in (0, \sqrt{n})$, the so-called L^2-range, which is easier, and then $\gamma \in [\sqrt{n}, \sqrt{2n})$, which uses more refined techniques.
- One could consider other, more general approximations of the field X rather than of the form $X * \theta_\varepsilon$ as well for the conclusion to hold

GMC measure. Construction.

- Let θ be a smooth mollifier. Set $X_\varepsilon := X * \theta_\varepsilon$, where X has a covariance kernel of log-type, and $\theta_\varepsilon := \frac{1}{\varepsilon^n}\theta\left(\frac{\cdot}{\varepsilon}\right)$. It can be shown that the random measures

$$M_{\varepsilon,\gamma}(dy) = e^{\gamma X_\varepsilon(y) - \frac{\gamma^2 \mathbb{E}[X_\varepsilon(y)^2]}{2}} \sigma(dy)$$

  converge in probability in the space of Radon measures towards a random measure M_γ.
- The random measure M_γ does not depend on the mollifier θ
- If $\sigma(dy) = f(y)\,dy$, with $f > 0$, the measure M_γ is different from 0 iff $\gamma < \sqrt{2n}$.
- The proof of this convergence is divided into two cases: $\gamma \in (0, \sqrt{n})$, the so-called L^2-range, which is easier, and then $\gamma \in [\sqrt{n}, \sqrt{2n})$, which uses more refined techniques.
- One could consider other, more general approximations of the field X rather than of the form $X * \theta_\varepsilon$ as well for the conclusion to hold

GMC measure. Properties.

For $\gamma < \sqrt{2n}$, assume $\sigma(dx) = f(x)\,dx$, with f bounded.

- Moments of GMC measure associated to LGF: For any ball $B \subset D$ we have $\mathbb{E}[M_\gamma(B)^q] < +\infty$ iff $q \in (-\infty, \frac{2n}{\gamma^2})$. Compare this with moments of Gaussians (i.e. Fernique thm)!
- Multifractal behavior of M_γ: Assume that f is continuous. Then for all x and all $q \in (-\infty, \frac{2n}{\gamma^2})$ there exists some positive constant $C_x = C_x(f, q, K)$ such that

$$\mathbb{E}[M_\gamma(B(x,r))^q] \overset{r \to 0}{\sim} C_x r^{\xi(q)},$$

where $\xi(q) = (n + \frac{\gamma^2}{2})q - \frac{\gamma^2 q^2}{2}$ is called the structure function of M_γ. Roughly speaking, the GMC measure is Hölder around each point.

GMC measure. Properties.

For $\gamma < \sqrt{2n}$, assume $\sigma(dx) = f(x)\,dx$, with f bounded.

- Moments of GMC measure associated to LGF: For any ball $B \subset D$ we have $\mathbb{E}[M_\gamma(B)^q] < +\infty$ iff $q \in (-\infty, \frac{2n}{\gamma^2})$. Compare this with moments of Gaussians (i.e. Fernique thm)!
- Multifractal behavior of M_γ: Assume that f is continuous. Then for all x and all $q \in (-\infty, \frac{2n}{\gamma^2})$ there exists some positive constant $C_x = C_x(f, q, K)$ such that

$$\mathbb{E}[M_\gamma(B(x,r))^q] \overset{r \to 0}{\sim} C_x r^{\xi(q)},$$

where $\xi(q) = (n + \frac{\gamma^2}{2})q - \frac{\gamma^2 q^2}{2}$ is called the structure function of M_γ. Roughly speaking, the GMC measure is Hölder around each point.

GMC measure. Properties.

For $\gamma < \sqrt{2n}$, assume $\sigma(dx) = f(x)\,dx$, with f bounded.

- Moments of GMC measure associated to LGF: For any ball $B \subset D$ we have $\mathbb{E}[M_\gamma(B)^q] < +\infty$ iff $q \in (-\infty, \frac{2n}{\gamma^2})$. Compare this with moments of Gaussians (i.e. Fernique thm)!
- Multifractal behavior of M_γ: Assume that f is continuous. Then for all x and all $q \in (-\infty, \frac{2n}{\gamma^2})$ there exists some positive constant $C_x = C_x(f, q, K)$ such that

$$\mathbb{E}[M_\gamma(B(x,r))^q] \overset{r \to 0}{\sim} C_x r^{\xi(q)},$$

where $\xi(q) = (n + \frac{\gamma^2}{2})q - \frac{\gamma^2 q^2}{2}$ is called the structure function of M_γ. Roughly speaking, the GMC measure is Hölder around each point.

References / recommended reading

General textbooks:

- *Brownian Motion: An Introduction to Stochastic Processes.* René L. Schilling.
- *Gaussian Hilbert Spaces.* Svante Janson.
- *Malliavin-laskenta: eli gaussisten prosessien derivointi.* Tommi Sottinen. Available freely on author's website.
- *Stochastic Partial Differential Equations: A Modeling, White Noise Functional Approach.* Helge Holden, Bernt Øksendal, Jan Ubøe, Tusheng Zhang.

Some more specific survey articles:

- Fractional Gaussian fields: a survey (2016). Asad Lodhia, Scott Sheffield, Xin Sun, Samuel S. Watson. In arXiv.
- Log-correlated Gaussian fields: an overview (2014). Bertrand Duplantier, Rémi Rhodes, Scott Sheffield, Vincent Vargas. In arXiv.
- Lecture notes on Gaussian multiplicative chaos and Liouville Quantum Gravity (2016). Rémi Rhodes, Vincent Vargas. In arXiv.

Thank you!