Geometry of quasiregular mappings and curves

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Outline

1. Quasiregular mappings
   - Between Euclidean spaces
   - Between Riemannian manifolds
   - Properties
2. Riemannian and lower dimensional volume forms
3. Quasiregular curves
   - Definition
   - Examples
   - Some results

Quasiregular mappings between Euclidean spaces

Definition

Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a domain. A mapping $f \colon \Omega \to \mathbb{R}^n$ is K-quasiregular for $K \geq 1$ if $f \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ and

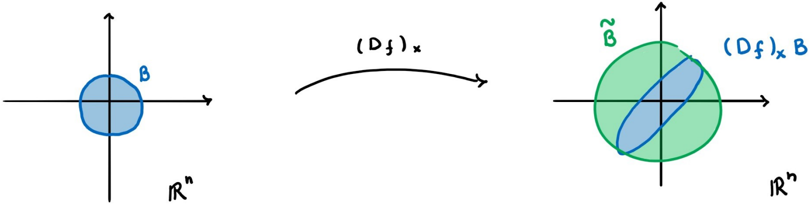
$$||Df||^n \leq K \det Df \text{ a.e. in } \Omega,$$

where $||Df||$ is the operator norm.

- The mapping f can be redefined in a set of measure zero so that it is made continuous.
- If f is a homeomorphism onto its image, then f is called quasiconformal.
- Every map $g \in W^{1,n}_{\text{loc}}(\Omega, \mathbb{R}^n)$ satisfies $\det Dg \leq ||Dg||^n$ a.e. in Ω.

Geometric interpretation

Let $f\colon \Omega \to \mathbb{R}^n$ be a non-constant K-quasiregular mapping. At almost every point $x \in \Omega$, the differential $(Df)_x$ behaves as follows:



In the picture, we have that

$$m_n(\tilde{B}) = ||(Df)_x||^n\, m_n(B) \text{ and } m_n((Df)_x B) = (\det(Df)_x) m_n(B).$$

Thus, the K-quasiregularity of f implies that $m_n(\tilde{B}) \leq K m_n((Df)_x B)$.

Connection to non-linear PDEs

Let $S(n)$ denote the space of symmetric positive definite $n \times n$-matrices having determinant equal to one. Given a domain $\Omega \subset \mathbb{R}^n$ and a bounded measurable mapping $G \colon \Omega \to S(n)$, the corresponding Beltrami system is defined as

$$(Df)^T Df = (\det Df)^{\frac{2}{n}} G \text{ a.e. in } \Omega$$

for mappings f with non-negative Jacobian determinant.

Quasiregular mappings are solutions to Beltrami systems and vice versa.

If $G \equiv I_n$, then the corresponding Beltrami system reduces to the Cauchy-Riemann system

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For $n = 2$, holomorphic mappings are solutions to the Cauchy-Riemann system and hence quasiregular.

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Examples

- Let $f: \Omega \to \mathbb{R}^n$ be L-Lipschitz continuous and assume that $\det Df \geq C$ for some constant $C > 0$. Then $||Df||^n \leq L^n \leq \frac{L^n}{C} \det Df$.
- Let k be a positive integer. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $(r, \varphi) \mapsto (r, k\varphi)$ in polar coordinates. For $(x, y) \neq 0$ we have that

$$(Df)_{(x,y)} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -1/r\sin\varphi & 1/r\cos\varphi \end{pmatrix}.$$

  It follows that $||Df|| = k$ and $\det Df = k$ in $\mathbb{R}^2 \setminus \{0\}$. Thus, f is k-quasiregular.
- Let ℓ be a positive integer and let $g: \mathbb{R}^2 \times \mathbb{R}^\ell \to \mathbb{R}^2 \times \mathbb{R}^\ell$ be the mapping $g = f \times \mathrm{id}_{\mathbb{R}^\ell}$. Then $||Dg|| = ||Df||$ and $\det Dg = \det Df$ in $(\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^\ell$. Hence g is $k^{\ell+1}$-quasiregular.

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Definition

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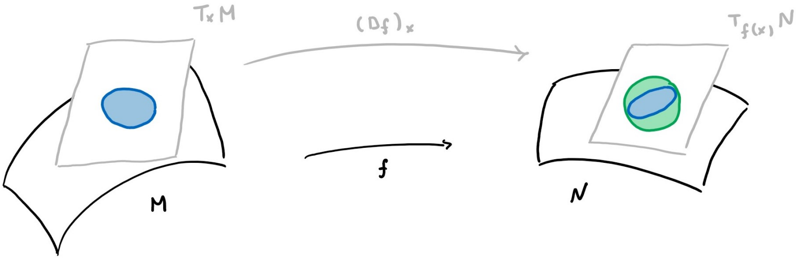

Classical results for quasiregular mappings

Let $f\colon M \to N$ be a non-constant quasiregular mapping between connected, oriented Riemannian n-manifolds. Then

- f is discrete and open, i.e., the preimage of each point is a discrete set and the image of each open set is open,
- there exists $p = p(n, K) > n$ for which $f \in W^{1,p}_{\mathrm{loc}}(M, N)$,
- f is differentiable a.e. in M, and
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Examples

- Smooth local isometries are 1-quasiregular. In particular, Riemannian covering maps are 1-quasiregular.
- Define $A\colon \mathbb{R}^2 \to \mathbb{S}^2$ as follows:
- Let $Z\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the mapping $(x, y, z) \mapsto e^z A(x, y)$.

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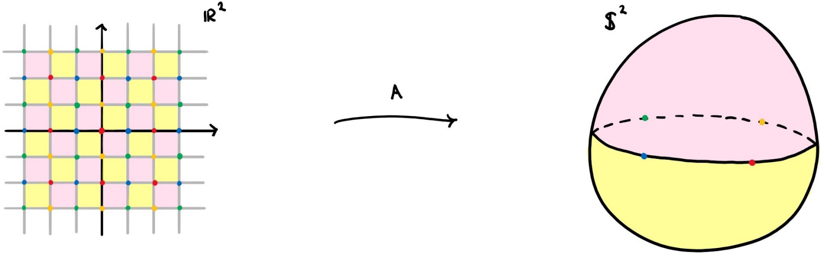

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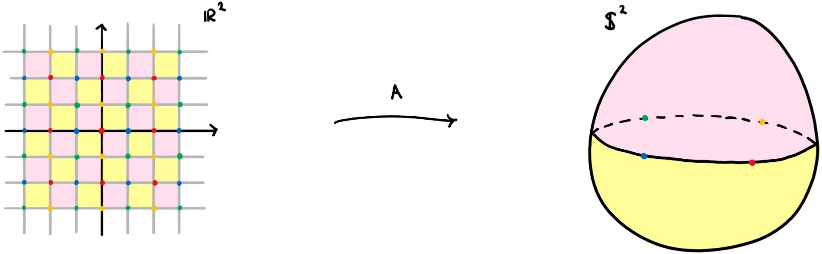

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Riemannian volume forms

Definition

A smooth n-manifold is orientable if there exists a smooth pointwise non-vanishing n-form $\omega \in \Omega^n(M)$. Such a form ω is called an orientation form on M and any basis $v_1, \ldots, v_n$ of a tangent space $T_x M$, $x \in M$, which satisfies $\omega_x(v_1, \ldots, v_n) > 0$ is called a positively oriented basis.

Definition

Let M be an oriented Riemannian n-manifold and let ω be an orientation form on M. If $\omega_x(v_1, \ldots, v_n) = 1$, for every $x \in M$ and for every positively oriented orthonormal basis $v_1, \ldots, v_n$ of $T_x M$, then we say that ω is the Riemannian volume form on M and we denote $\omega = \mathrm{vol}_M$.

Proposition

The Riemannian volume form vol_M exists for each oriented Riemannian manifold M.

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Jacobian determinant on Riemannian manifolds

Let M and N be connected, oriented Riemannian n-manifolds and let $f\colon M \to N$ be quasiregular. Then $||Df||^n \leq K \det Df$ a.e. in M.

...BUT! How is $\det Df$ formally defined?

For a.e. every $x \in M$, we have that $(Df)_x$ is a linear map $T_xM \to T_{f(x)}N$. Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ be positively oriented orthonormal bases of T_xM and $T_{f(x)}N$, respectively. Then $(Df)_x$ has a matrix representation, denoted A_x, with respect to the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$. Now

$$\det(Df)_x = \det A_x.$$

Defining $\det Df$ pointwise a.e. as above yields that

$$f^* \operatorname{vol}_N = (\det Df) \operatorname{vol}_M .$$

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Lower dimensional volume forms

The forms $dx \wedge dy$, $3\,dy \wedge dz$, and $e^x\,dy \wedge dz$ are examples of smooth pointwise non-vanishing 2-forms on $\mathbb{R}^3$.

Definition

A smooth n-form $\omega \in \Omega^n(N)$ on a Riemannian m-manifold N, where $n \leq m$, is an n-volume form if ω is pointwise non-vanishing and $d\omega = 0$, where d denotes the exterior derivative $d\colon \Omega^n(N) \to \Omega^{n+1}(N)$.

- $d(dx \wedge dy) = 0$, so $dx \wedge dy$ is a 2-volume form on $\mathbb{R}^3$
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Quasiregular curves

Definition

A continuous mapping $F\colon M \to N$ between connected, oriented Riemannian manifolds, $2 \le n = \dim M \le \dim N$, is a K-quasiregular ω-curve for $K \ge 1$ with respect to an n-volume form $\omega \in \Omega^n(N)$ if $F \in W^{1,n}_{\mathrm{loc}}(M,N)$ and

$$(||\omega|| \circ F)\, ||DF||^n \le K(\star F^*\omega) \text{ a.e. in } M.$$

Here $||\omega||$ is the comass norm given pointwise by

$$||\omega_x|| = \max\{\omega_x(v_1,\ldots,v_n)\colon v_1,\ldots,v_n \in T_xN \text{ unit vectors}\}$$

and the function $\star F^*\omega$ is determined by the equation

$$F^*\omega = (\star F^*\omega)\,\mathrm{vol}_M\,.$$

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- When $n = m$ and $\omega = \mathrm{vol}_N$, the definition reduces to the definition of quasiregular mappings.
- Every $G \in W^{1,n}_{\mathrm{loc}}(M, N)$ satisfies $\star G^*\omega \leq (||\omega|| \circ G)\, ||DG||^n$ a.e. in M.

An example and its geometric interpretation

Let $\omega = dx \wedge dy \in \Omega^2(\mathbb{R}^3)$ and let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a K-quasiregular ω-curve. Write $F = (f_1, f_2, f_3)$. Let $F'\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $F' = (f_1, f_2)$.

Then $||DF|| \geq ||DF'||$ and $\star F^*\omega = \det DF'$.

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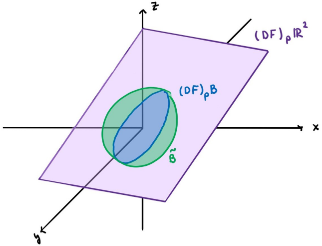
How $(DF)_p$ maps the unit ball:

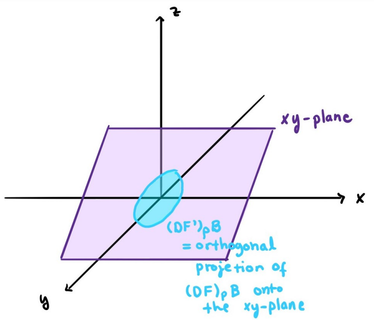


Let $\lambda_1 \geq \lambda_2$ be the singular values of $(DF)_p$. Then $\mathcal{H}^2(\tilde{B}) = \lambda_1^2 \mathcal{H}^2(B) = ||DF||^2 \mathcal{H}^2(B)$ and $\mathcal{H}^2((DF)_p B) = \lambda_1 \lambda_2 \mathcal{H}^2(B)$. In general, $\lambda_1\lambda_2 \neq \star F^*\omega$.

An example and its geometric interpretation

Let $\omega = dx \wedge dy \in \Omega^2(\mathbb{R}^3)$ and let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a K-quasiregular ω-curve. Write $F = (f_1, f_2, f_3)$. Let $F'\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping $F' = (f_1, f_2)$.

How $(DF')_p$ maps the unit ball:



Let $\mu_1 \geq \mu_2$ be the singular values of $(DF')_p$. Then
$\mathcal{H}^2((DF')_p B) = \mu_1 \mu_2 \mathcal{H}^2(B) = (\det DF')\mathcal{H}^2(B) = (\star F^*\omega)\mathcal{H}^2(B)$.

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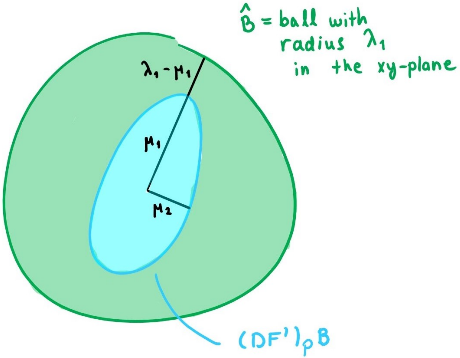
Comparing the two:



Now the inequality $||DF||^2 \leq K(\star F^*\omega)$ yields that $\mathcal{H}^2(\hat{B}) \leq K\mathcal{H}^2((DF')_p B)$.

Examples

- Let also M be a connected, oriented Riemannian n-manifold and let $N = N_1 \times \cdots \times N_k$ be a Riemannian product of oriented Riemannian n-manifolds. Let also $f_i \colon M \to N_i$ be quasiregular mappings. Then $F = (f_1, \ldots, f_k) \colon M \to N$ is a quasiregular $\mathrm{vol}_N^\times$-curve, where $\mathrm{vol}_N^\times$ is the n-volume form obtained from the product structure of N.
- Let $\Omega \subset \mathbb{C}$ be a domain and let $F \colon \Omega \to \mathbb{C}^k$ be a holomorphic curve. Then F is a quasiregular ω_{sym}-curve, where ω_{sym} is the standard symplectic form on $\mathbb{C}^k$.
- Fix $y \in \mathbb{R}^n$ and define $F \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ by $x \mapsto (x, x \cdot y)$. Then F is a quasiregular $dx_1 \wedge \cdots \wedge dx_n$-curve.
- Let $p \colon \mathbb{R}^{n+1} \to T^{n+1}$ be the standard (Riemannian) covering map and let $\omega = \pi_1^* \mathrm{vol}_{\mathbb{S}^1} \wedge \cdots \wedge \pi_n^* \mathrm{vol}_{\mathbb{S}^1}$, where each $\pi_i \colon T^{n+1} \to \mathbb{S}^1$ is the projection. Then $p \circ F \colon \mathbb{R}^n \to T^{n+1}$ is a quasiregular ω-curve.

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A different kind of example

Let $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the mapping $(x, y) \mapsto (y, x, -2x)$. Let $\omega_1 = dx \wedge dy \in \Omega^2(\mathbb{R}^3)$ and $\omega_2 = dx \wedge dy + dx \wedge dz \in \Omega^2(\mathbb{R}^3)$.

Then $||DF|| = \sqrt{5}$, $\star F^*\omega_1 = -1$, and $\star F^*\omega_2 = 1$. We also have that $||\omega_1|| = 1$ and $||\omega_2|| = \sqrt{2}$.

We want to know if there exists $K_i \geq 1$ satisfying

$$(||\omega_i|| \circ F)\, ||DF||^2 \leq K_i(\star F^*\omega_i)$$

for $i = 1, 2$.

Since $\star F^*\omega_1 < 0$, no suitable K_1 exists. For K_2 we can choose $K_2 = 5\sqrt{2}$.

Hence, F is not a quasiregular ω_1-curve but it is a quasiregular ω_2-curve.

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Some results

Let $F\colon M \to N$ be a non-constant K-quasiregular ω-curve between connected, oriented Riemannian manifolds with $n = \dim M < \dim N$.

- Open?
- Discrete?

- $F \in W^{1,p}_{\text{loc}}(M, N)$ for some $p = p(n, K) > n$?
- Differentiable a.e.?
- $\star F^* \omega > 0$ a.e.?

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Thank you!