Renormalization Group approach to Stochastic Quantization

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Renormalization and Renormalization Group

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Renormalization and Renormalization Group

- Renormalization Group is a tool for studying how physics behaves on different scales. The interesting question is how does this framework handle the problem of physics breaking down in small scales where any interaction proportional to $\frac{1}{r^2}$ becomes infinitely strong.
- We begin by looking at these infinities in more detail and how they they are cancelled out in a process called Renormalization.

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Renormalization

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- If the bare electron shell has a negative mass we might be able to take the point electron limit in such a way that the infinite negative bare mass and infinite positive Coulomb potential energy cancel out leaving a finite result.

Renormalization

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- Quantum mechanical case seems to be worse: In addition to the infinite self-energy virtual particles are constantly being created and annihilated around the electron affecting its properties. These contributions sum up to infinity, as calculated by Oppenheimer in 1929 [**1**].
- This problem was then independently solved by Feynman, Schwinger and Tomonaga and their solutions combined and summarized by Dyson in 1948 [**2**]: The key insight was to realize that while the theoretical electron exists as a point particle, real-world measurements would see not only the point electron but also the cloud of virtual particles surrounding it.

Renormalization

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- "...the present treatment should be regarded as justified by its success in applications rather than by its theoretical derivation." -Dyson, F.J. 1948 [**2**]

Renormalization Group

- This insight that the properties of an electron depend on the scale it is being observed is essentially a Renormalization Group idea. While Feynman, Schwinger, Tomonaga and Dyson used it to make sense of real-world measurements, Gell-Mann and Low made the the Renormalization Group approach more explicit in 1954 and asked how the mathematics would behave if we approached arbitrarily small scales [**3**].

Renormalization Group

- Assume we start with a lattice model. To look at a larger scale, Renormalization Group consists of two operations. First course graining: We average out the nearby lattice points into a new lattice point. Then scaling: We rescale this new system into the same level of detail. We can then repeat this process until the desired scale is reached.

Renormalization Group

- Assume we start with a lattice model. To look at a larger scale, Renormalization Group consists of two operations. First course graining: We average out the nearby lattice points into a new lattice point. Then scaling: We rescale this new system into the same level of detail. We can then repeat this process until the desired scale is reached.
- The challenge here lies in accessing small scales as this corresponds to setting the initial lattice spacing to zero. This means we need to iterate infinite number of times to get to any finite scale. Any term that grows with each iteration step would then become infinite.

Stochastic Quantization

- Stochastic Quantization is a way of describing quantum mechanics with stochastic partial differential equations constructed with Brownian motion and Gaussian measures, $\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2)$.
- Renormalization and Renormalization Group are used both in Stochastic Quantization and usual approaches.

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Stochastic Quantization

- Stochastic Quantization began with two derivations of Schrödinger equation using stochastic Newtonian mechanics in the 1960's by Nelson and De La Peña-Auerbach [**4**][**5**].
- These derivations replaced the wave amplitude with the probability density. Uncertainties associated with quantum mechanics were seen as a result of a random Gaussian force. Schrödinger equation can be recovered with the use of continuity equation for the probability density $\frac{\partial \rho}{\partial t} + (\nabla j) = 0$ and writing terms in specific ways.

Stochastic Quantization

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Stochastic Quantization

- "We shall attempt to show in this paper that the radical departure from classical physics produced by the introduction of quantum mechanics forty years ago was unnecessary. An entirely classical derivation and interpretation of the Schrödinger equation will be given..." -Nelson, E, 1966 [**4**]
- In the modern path integral formalism of quantum mechanics we integrate over all possible paths and need a measure that assigns a weight to each path. Stochastic Quantization is the process of recovering that measure from a stochastic partial differential equation.

- Let's look at the Schrödinger equation of a free particle

$$i\partial_t\psi(t,x) = -\frac{1}{2}\Delta\psi(t,x) = H_0\psi(t,x).$$

This equation has the time-evolution given by

$$\psi(t,x) = (e^{-itH_0}\psi_0)(x).$$

- In the path integral formalism this is written as

$$(e^{-itH_0}\psi_0)(x) = \int_{\mathbb{R}^d} K_0(t,x,y)\psi_0(y)dy$$

where $K_0(t,x,y) = (2\pi it)^{-d/2}e^{i\frac{(x-y)^2}{2t}}$ is a Gaussian measure in imaginary time!

Stochastic Quantization

- We have shown that free quantum mechanical particle is described by a Gaussian measure in imaginary time. Next we will recover Gaussian measures from stochastic partial differential equations.
- Let A be positive, symmetric $d \times d$ -matrix, $A_{ik} > 0, A^T = A$ and $X(t)$ d-dimensional random variable satisfying the equation

$$\dot{X}(t) = -kAX(t) + \sqrt{2k}\xi(t).$$

- Here $\xi(t)$ is a white noise process which can be thought of as a time-derivative of Brownian motion. It satisfies

$$\mathbb{E}[\xi_i(t)\xi_j(s)] = \delta_{ij}\delta(t-s).$$

Stochastic Quantization

- There are two ways to recover a Gaussian measure from this stochastic partial differential equation. Both are related to the idea of a system described by the equation eventually settling down in a certain state.
- The invariant measure, where if $X(t)$ has a certain initial probability distribution it also has that same probability distribution at a later time, is a Gaussian measure $e^{-\frac{1}{2}(x,Ax)}$.
- The $t \to \infty$ limit of $X(t)$ is a Gaussian random variable with probability measure given by $e^{-\frac{1}{2}(x,Ax)}$.

Renormalization Group Calculation

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- Next introduce a nonlinear interaction term to a stochastic partial differential equation. Setting this interaction term to zero corresponds to the Gaussian case of a free quantum mechanical particle we saw earlier.
- We then wish to apply the Renormalization Group ideas of averaging and rescaling to this equation and see how the interaction term changes under this transformation.
- We will show that the interaction term is finite in finite scales if and only if it becomes infinite in arbitrarily small scales. Think of the result that the mass of an electron appears finite to us if and only if the mass of a theoretical point electron is infinite.

Renormalization Group Calculation

- The quantum field ϕ satisfies the following equation

$$\dot{\phi} = \Delta_N \phi - \lambda_N \phi^3 - r_N \phi + \xi_N$$

where $v^{(N)} = \lambda_N \phi^3 + r_N \phi$ is an interaction term.

- Here the index N corresponds to **regularization**: We define the equation on a lattice where distance between lattice points and therefore the smallest allowed scale is L^{-N}. This is necessary because ξ isn't smooth enough. We then wish to take the $N \to \infty$ limit. For the following calculation we assume that the system has been rescaled to have a lattice spacing of 1.

Renormalization Group Calculation

We define an operator that averages an L^n-lattice cube

$$(\mathcal{P}_n\phi)(x) = L^{-nd} \sum_{[L^{-n}y]=[L^{-n}x]} \phi(y).$$

This lets us decompose the field as follows

$$\phi = \mathcal{P}_1\phi + (1-\mathcal{P}_1)\phi := s_L\phi' + z$$

$$(s_L\phi')(t,x) = (\mathcal{P}\phi)(t,x) = L^{\frac{2-d}{2}}\phi'(L^{-2}t,[L^{-1}x]).$$

The factors here ensure that we will recover the original equation. Here we have the averaged and rescaled field that corresponds to a larger scale $L^{-(N-1)}$

$$\phi'(t,x) := L^{-\frac{2+d}{2}} \sum_{[\frac{u}{L}]=0} \phi(L^2t, Lx+u).$$

Renormalization Group Calculation

Note that by construction ϕ' is constant in L-cubes and z has zero average over them. The lattice Laplacian in our equation is defined as

$$-\Delta_N = \sum_{n=0}^{N-1} (\mathcal{P}_n - \mathcal{P}_{n+1}) L^{-2n}.$$

The decomposed field satisfies

$$\begin{aligned} \Delta_N(s_L \phi') &= L^{-2} s_L(\Delta_{N-1}\phi') \\ \Delta_N(z) &= -z. \end{aligned}$$

Renormalization Group Calculation

We now decompose the entire equation into $\mathcal{P}_1$ and $1 - \mathcal{P}_1$ -parts just like we did with the field

$$\dot{\phi}' = \Delta_{N-1}\phi' - w + \xi'$$
$$\dot{z} = -z - u + \xi_{\perp}.$$

We are interested in the equation for ϕ' and particularly in the new interaction term w. However it depends on z so the second equation would also need to be solved.

Renormalization Group Calculation

New interaction term is calculated in the same way as the new field, by averaging the cube

$$w(t,x) = L^{\frac{2-d}{2}} \sum_{[\frac{y}{L}]=x} v^{(N)}(L^2 t, y; s_L \phi' + z).$$

This calculation gives

$$\begin{aligned} &L^{\frac{2-d}{2}} \sum_{[\frac{u}{L}]=0} \left(\lambda_N (L^{\frac{2-d}{2}} \phi' + z)^3 + r_N (L^{\frac{2-d}{2}} \phi' + z) \right) \\ =&L^{4-d} \lambda_N \phi'^3 + (L^2 r_N + \rho_1 \lambda_N)\phi' + \rho_2 \end{aligned}$$

where ρ_1, ρ_2 are some terms that depend on z but not on ϕ'.
Remember that by our construction
$\sum_{[\frac{u}{L}]=0} \phi' = L^d \phi'$, $\sum_{[\frac{u}{L}]=0} z = 0$.

Renormalization Group Calculation

This lets us express the new factors of ϕ^3 and ϕ, which we denote by λ_{N-1}, r_{N-1}, in terms of the old factors λ_N, r_N

$$\lambda_{N-1} = L^{4-d}\lambda_N$$
$$r_{N-1} = L^2 r_N + \rho_1 \lambda_N.$$

New factors correspond to the interaction term on scale $L^{-(N-1)}$. We then wish to calculate the factors corresponding to some predetermined scale, λ_n, r_n. For these we need to repeat our iteration $N - n$ times. The challenge is to, for each N set the initial condition λ_N, r_N in such a way that we can take the $N \to \infty$ limit and calculate finite λ_n, r_n.

Renormalization Group Calculation

We now set $d = 2$. We define new variables $r_n = L^{-2n} h_n, \lambda_n = L^{-2n} g_n$. These then satisfy the iteration

$$\begin{aligned} g_{N-1} &= g_N \\ h_{N-1} &= h_N + L^{-2} \rho_1 g_N. \end{aligned}$$

From this we can easily calculate

$$h_n = h_N + (N-n) L^{-2} \rho_1 g_N$$

which diverges as $N \to \infty$ and to cancel it we set the initial condition

$$\begin{aligned} h_N &= -L^{-2} \rho_1 N g_N \\ r_N &= -L^{-2} \rho_1 N \lambda_N. \end{aligned}$$

Renormalization Group Calculation

With this we get the result

$$\lambda_n = L^{2(N-n)}\lambda_N = L^{-2n}\lambda$$
$$r_n = -nL^{2(N-n)}\rho_1\lambda_N = -nL^{-2n}\rho_1\lambda.$$

- Here we choose some small constant λ and set the second initial condition λ_N to satisfy $\lambda = L^{2N}\lambda_N$.
- The parameters λ_n and r_n are now independent of N and therefore remain finite. This is true if and only if we set the initial condition r_N to become infinite in the for $N \to \infty$ limit.

Thank you for your attention!

References

[1] Oppenheimer, J.R. (1930). Note on the Theory of the Interaction of Field and Matter, Phys. Rev. 35, 461

[2] Dyson, F.J. (1949). The Radiation Theories of Tomonaga, Schwinger and Feynman, Phys. Rev 75, 486.

[3] Gell-Mann, M. and Low, F. E. (1954). Quantum Electrodynamics at Small Distances, Physical Review Volume 96, Number 5, pages 1300-1312.

[4] Nelson, E. (1966). Derivation of the Schrödinger Equation from Newtonian Mechanics, Physical Review Volume 150, Number 4, pages 1079-1085.

[5] De La Peña-Auerbach, L. (1967). A Simple Derivation of the Schoedinger Equation from the Theory of Markoff Processes, Physics Letters Volume 24A. Number 11, pages 603-604.

Appendix A: Derivation of the invariant measure

Let $x(t)$ be a random vector and $\omega(x)$ be a vector-valued function, $x(t) \in \mathbb{R}^d, \quad \omega(x) : \mathbb{R}^d \to \mathbb{R}^d$ and $k > 0$. Now consider the group of stochastic differential equations

$$dx_i(t) = \omega_i(x)dt + \sqrt{2k}dB_i(t) \quad i = 1, .., d$$

where B_i are independent Brownian motions with expected value 0 and covariance structure given by

$$\mathbb{E}[B_i(t)B_j(t)] = \delta_{ij}(t \wedge s).$$

We now look for invariant measure $\varphi(x)dx$ such that if we have $x(0) \overset{law}{=} \varphi(x)dx$ then for an arbitrary later moment t we would also have $x(t) \overset{law}{=} \varphi(x)dx$.

Appendix A: Derivation of the invariant measure

This means that $x(t)$ is stationary. So for any sufficiently smooth function $F(x)$ we have

$$\int \mathbb{E}(F(x) \mid x(0) = x)\varphi(x)dx = \int F(y)\varphi(y)dy.$$

The Itô lemma states

$$dF(x) = \sum_{i=1}^{d} \frac{\partial F(x)}{\partial x_i} dx_i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 F(x)}{\partial x_i \partial x_j} dx_i dx_j$$

Applying it to $F(x)$ we get

$$\begin{aligned} dF(x) &= \sum_{i=1}^{d} \left[\omega_i(x)dt + \sqrt{2\kappa}dB_i(t)\right] \frac{\partial F(x)}{\partial x_i} + k\frac{\partial^2 F(x)}{\partial x_i^2} dt \\ &= [\omega(x)dt + \sqrt{2k}dB(t)] \cdot \nabla F(x) + k\Delta F(x)dt. \end{aligned}$$

Appendix A: Derivation of the invariant measure

Invariant measure $\varphi(x)dx$ does not depend on time

$$0 = \frac{d}{dt}\mathbb{E}[F(x)] = \int \Big[\omega(y) \cdot \nabla F(y) + k\Delta F(y)\Big]\varphi(y)dy.$$

Here the $\sqrt{2k}dB_i(t)$ term vanishes when taking the expected value because $\mathbb{E}[dB_i(t)] = 0$. Next we will use the multidimensional integration by parts formula

$$\int_C u(y) \cdot \nabla v(y)dy = \int_{\partial C} u(y)v(y) \cdot n_i dS - \int_C \nabla u(y) \cdot v(y)dy.$$

Appendix A: Derivation of the invariant measure

The boundary term vanishes. Applying this formula once to the first term and twice to the second term we get

$$0 = \int F(x)\Big[k\Delta\varphi(y) - \nabla\cdot\omega(y)\varphi(y)\Big]dy.$$

Since this is true for an arbitrary function $F(x)$ then for invariant measure we have

$$\nabla\cdot\Big[k\nabla\varphi(x) - \omega(x)\varphi(x)\Big] = 0.$$

Inserting $\omega(x) = kAx(t)$ into this gives our result and the invariant measure $e^{-\frac{1}{2}(x,Ax)}$.

Appendix B: Derivation of distribution

Let us consider a linear equation with $\omega(x) = kAx$, where A is a positive, symmetric $d \times d$ -matrix, $A_{ij} > 0, A^T = A$ and $x(t)$ is a vector, $x(t) \in \mathbb{R}^d$

$$\dot{x}(t) = -kAx(t) + \sqrt{2k}\dot{B}(t).$$

We can solve this by applying the Itô lemma to $f(t,x) = e^{kAt}x(t)$

$$\begin{aligned}
df(t,x) &= \frac{\partial f(t,x)}{\partial x}dx + \frac{\partial f(t,x)}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f(t,x)}{\partial x^2}(dx)^2 \\
&= e^{kAt}dx - kAe^{kAt}x(t)dt \\
&= e^{kAt}[kAx(t)dt + \sqrt{2k}dB(t)] - kAe^{kAt}x(t)dt \\
&= e^{kAt}\sqrt{2k}dB(t).
\end{aligned}$$

Appendix B: Derivation of distribution

Taking the integral form and multiplying both sides by e^{-kAt}

$$\begin{aligned} x(t) &= e^{-kAt}x(0) + \sqrt{2k}\int_0^t e^{-kA(t-s)}dB(s) \\ &:= e^{-kAt}x(0) + \varphi(t). \end{aligned}$$

Here $\varphi(t)$ are Gaussian random variables. Let $t \geq t'$

$$\begin{aligned} \mathbb{E}[\varphi_i(t)\varphi_j(t')] = 2k\mathbb{E}\Big[&\sum_{l=1}^{d}\int_0^t [e^{-kA(t-s)}]_{il}dB_l(s) \\ &\times \sum_{k=1}^{d}\int_0^{t'} [e^{-kA(t'-s')}]_{jk}dB_k(s')\Big]. \end{aligned}$$

Appendix B: Derivation of distribution

Because we have $\mathbb{E}[dB_l(s)dB_k(s')] = \delta_{lk}\delta(s-s')ds$ the cross terms vanish and the expected value equals

$$
\begin{aligned}
&= 2k\Big[\sum_{l=1}^{d}\int_0^{t'} [e^{-kA(t-s')}]_{il} \times [e^{-kA(t'-s')}]_{jl}\Big] ds' \\
&= 2k\Big[\int_0^{t'} (e^{-kA(t-s')}) \times (e^{-kA^T(t'-s')})\Big]_{ij} ds' \\
&= 2k\Big[e^{-kA(t-t')}\int_0^{t'} e^{-2kA(t'-s')}\Big]_{ij} ds' \\
&= \Big[e^{-kA(t-t')}\Big(\frac{1-e^{-2kAt'}}{A}\Big)\Big]_{ij}.
\end{aligned}
$$

Appendix B: Derivation of distribution

Thus, as $t' \to \infty$ we have

$$\lim_{t' \to \infty} \mathbb{E}\varphi_i(t' + \tau)\varphi_j(t') = [e^{-\tau k A} A^{-1}]_{ij}.$$

Where now with $\tau = 0$ in the infinite time limit $x(t) \underset{t' \to \infty}{\to} x$ becomes a Gaussian random vector with covariance $\mathbb{E}[x_i x_j] = [A^{-1}]_{ij}$ and, indeed law

$$(x) = c \cdot exp(-\frac{1}{2}(x, Ax))dx.$$

Appendix C: Derivation of field decomposition

Here note that the scaling operation also affects $\mathcal{P}_n$.

$$\Delta_N(s_L\phi') = -\sum_{n=0}^{N-1}(\mathcal{P}_n - \mathcal{P}_{n+1})L^{-2n}L^{-\frac{d-2}{2}}\phi'(L^{-2}t, L^{-1}x)$$

$$= s_L\big[-\sum_{n=1}^{N-1}(\mathcal{P}_{n-1} - \mathcal{P}_n)L^{-2(n-1)}L^{-2}\phi'(t,x)\big]$$

$$= L^{-2}s_L\big[-\sum_{n=0}^{N-2}(\mathcal{P}_n - \mathcal{P}_{n+1})L^{-2n}\phi'(t,x)\big] = L^{-2}s_L(\Delta_{N-1}\phi')$$

Appendix C: Derivation of field decomposition

Here we use the result that follows from $\mathcal{P}_n\mathcal{P}_m = \mathcal{P}_n$ for $n \geq m$. Now we have

$$\begin{aligned}&(1-\mathcal{P}_1)(\mathcal{P}_n - \mathcal{P}_{n+1})\\ =&\mathcal{P}_n - \mathcal{P}_1\mathcal{P}_n - \mathcal{P}_{n+1} + \mathcal{P}_1\mathcal{P}_{n+1}\end{aligned}$$

where now for $n = 0$ this results in $1 - \mathcal{P}_1$ and for any $n > 0$ it equals 0. This gives our result

$$\Delta_N z = -\sum_{n=0}^{N-1}(\mathcal{P}_n - \mathcal{P}_{n+1})L^{-2n}(1-\mathcal{P}_1)\phi = -z.$$