Exponential functionals of stochastic processes with applications

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Outline

Pre-registered goals of the talk:

- Promote interest in exponential functionals and Marc Yor's research.
- Practice some of Beamer's features to build presentation skills.
- Calibrate predicted length of a presentation (plan: 30 minutes).

Problem 1: disordered systems

In statistical physics, a system with state space S described by Gibbs measures

$$\mathcal{G}(dx) = \frac{e^{-\beta H(x)}\,\mathrm{d}x}{\int_S e^{-\beta H(y)}\,\mathrm{d}y}.$$

Thermodynamic quantities of interest to the physics community tend to rely on computation of the *partition function*

$$Z = \int_S e^{-\beta H(y)}\,\mathrm{d}y,$$

or the *free energy* $\log Z$.
Disordered systems include some inhomogeneity in the Hamiltonian H.
An example would be random walk vs. the Sinai-Kesten random walk.

A continuous model

A different related toy model is that of a particle restricted to the interval $[0, L]$ and subject to a random force F. To model random inhomogeneities we assume that $F(x)$ is distributed as white noise around some mean value f_0. Then the potential U is

$$U(x) = -\int_0^x F(x)\,dx = f_0 x + \sigma B_x,$$

with B being a standard Brownian motion. For the sake of simplicity we use scaling properties to get rid of the inverse temperature β, and index with the drift,

$$Z_L = \int_0^L e^{-\beta U(x)}\,dx.$$

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$$Z_L^{(\mu,\sigma)} = \int_0^L e^{-\mu x - \sigma B_x}\,dx.$$

Since thermodynamic variables have the form of expectation values, the partition function must be computed:

$$g(x) = \frac{\int_0^L g(x) e^{-\beta U(x)}\ \mathrm{d}x}{\int_0^L e^{-\beta U(x)}\ \mathrm{d}x}.$$

Special case: zero drift and L exponential random variable

We define a related functional

$$A_t^{(\nu)} = \int_0^t e^{2(\nu s + B_s)}\, ds$$

which can be related to Z through scaling. In particular, we write $A_t = A_t^{(0)}$.
An application of Bougerol's identity

$$\sinh(B_t) \stackrel{d}{=} W_{A_t},$$

where W Brownian motion independent of the B contained in A, allows recovery of much information about the distribution of A_t. We focus first on the case with $L = L_\lambda$ an exponentially distributed random variable with parameter $\lambda > 0$.

Bougerol's identity

Theorem (Alili and Dufresne)

Let B_t, W_t be two independent standard Brownian motions, and let $A_t = \int_0^t e^{2B_s}\,\mathrm{d}s$. Then, it holds that

$$\sinh(B_t) \stackrel{d}{=} W_{A_t}.$$

Sketch of argument.

Consider the Markov process

$$X_t = e^{B_t} \int_0^t e^{-B_s}\,\mathrm{d}W_s, \quad W \perp B.$$

An application of Itô's formula yields the SDE

$$\mathrm{d}X_t = \tfrac{1}{2} X_t\,\mathrm{d}t + (X_t\,\mathrm{d}B_t + \mathrm{d}W_t).$$

The above can equivalently be expressed as

$$\mathrm{d}X_t = \tfrac{1}{2} X_t\,\mathrm{d}t + (X_t^2 + 1)^{1/2}\,\mathrm{d}\beta_t,$$

with β_t another Brownian motion.

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Sketch of argument.

An application of Itô's formula to the process $Y_t = \sinh(\beta_t)$ yields

$$\mathrm{d}Y_t = \frac{1}{2}Y_t\,\mathrm{d}t + (Y_t\,\mathrm{d}B_t + \mathrm{d}W_t).$$

A simple inspection lets us conclude that $X_t \overset{d}{=} Y_t$, for any fixed $t \geq 0$. Finally, conditioning on the process B and applying the Itô isometry allows us to conclude

$$X_t \overset{d}{=} \int_0^t e^{2B_s}\,\mathrm{d}s,$$

and the identity now follows. □

Special case: zero drift and L exponential random variable

We observe first that by Brownian scaling

$$\mathbb{E}[W_{A_{L_\lambda}}^{2m}] = \mathbb{E}[N^{2m}]\mathbb{E}[A_{L_\lambda}^m], \tag{1}$$

where N is a standard normal r.v.

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Furthermore, $|B_{L_\lambda}|$ is distributed as an exponential variable with parameter $\theta = \sqrt{2\lambda}$. As such,

$$\mathbb{E}[\sinh(B_{L_\lambda})^{2m}] = \int_0^\infty \theta e^{-\theta x}(\sinh x)^{2m}\, \mathrm{d}x.$$

Provided that $\theta > 2m$, an application of the formula $\sinh x = \frac{1-e^{-2x}}{2e^{-x}}$ and a change of variables $x = -\frac{1}{2}\log t$ yields the formula

$$\mathbb{E}[\sinh(B_{L_\lambda})^{2m}] = \frac{\theta}{2^{2m}} B\left(\frac{\theta - 2m}{2}, 2m+1\right), \tag{2}$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t$ is the Beta function.

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The use of (1)-(2) with an application of Bougerol's identity $\sinh(B_t) \overset{d}{=} W_{A_t}$, allows us to obtain the formula, valid for $m \geq 0$, $\theta > 2m$,

$$\mathbb{E}[A_{L_\lambda}^m] = \frac{\Gamma(1+m)\Gamma(\theta/2+1)\Gamma(\theta/2-m)}{2^m\Gamma(\theta/2)\Gamma(1+m+\theta/2)}.$$

It can be inferred from the formula

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that there are components of a beta and gamma variable, independently of each other. The following theorem holds.

Theorem

Let $\lambda > 0$, $a = \sqrt{\lambda/2}$, and let L_λ be an exponential variable with parameter λ, independent of the BM B. Then we have the identity in distribution

$$A_{L_\lambda} \stackrel{d}{=} \frac{\beta_{1,a}}{2\Gamma_a}, \quad \beta_{1,a} \perp \Gamma_a, \tag{3}$$

where $\beta_{1,a}$ denotes a beta variable and Γ_a a gamma variable, i.e.

$$\mathbb{P}(\beta_{1,a} \in dx) = a(1-x)^{a-1}1_{(0,1)}(x)dx,$$

$$\mathbb{P}(\Gamma_a \in dx) = \frac{x^{a-1}e^{-x}}{\Gamma(a)}1_{(0,\infty)}(x)dx.$$

The identity $A_{L_\lambda} \overset{d}{=} \frac{\beta_{1,a}}{2\Gamma_a}$ allows us to compute the mean free energy that we were originally interested in. It is the case that

$$\mathbb{E}[\log \Gamma_a] = \psi(a), \quad \mathbb{E}[\log \beta_{1,a}] = \psi(1) - \psi(1+a),$$

where $\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z} \log \Gamma(z)$ is the digamma function.

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$$\mathbb{E}[\log A_{L_\lambda}] = -\gamma - \frac{1}{\sqrt{\lambda/2}} - 2\psi(\sqrt{\lambda/2}).$$

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If we use Brownian scaling to move back to the original problem, we get the following.

Proposition

Let a particle be restricted to the interval $[0, L]$, L being exponential with parameter $\lambda > 0$, and let the particle be subject to the Brownian potential $U(x) = \sigma B_x$, B_x a standard Brownian motion. Then, the partition function and mean free energy of the system is

$$Z_L^{(0,\sigma)} = \int_0^L e^{-\sigma B_x}\,\mathrm{d}x, \quad \mathbb{E}[log Z_L^{(0,\sigma)}] = -2\log\sigma - \gamma - \frac{1}{\sigma}\sqrt{2/\lambda} - 2\psi(\sigma\sqrt{\lambda/2}).$$

Above γ denotes the Euler-Mascheroni constant $\gamma := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n$.

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2. Marc Yor has proved with another method that there is a more general identity

$$A^{(\nu)}_{T_\lambda} \stackrel{d}{=} \frac{\beta_{1,a}}{2\Gamma_b}, \quad a = \frac{1}{2}(\sqrt{\nu^2 + 2\lambda} + \nu), b = \frac{1}{2}(\sqrt{\nu^2 + 2\lambda} - \nu).$$

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These naturally lead to a whole family of explicitly solvable models with drifted potentials.
3. If $\nu > 0$ and we let $\lambda \to 0$ in the identity above, we obtain the result
$$A_\infty^{(-\nu)} = \frac{1}{2\Gamma_\nu}.$$
This was first proved by D. Dufresne in an insurance setting.
4. Knowledge of $\mathbb{E}[A_{T_\lambda}^{(\nu)}]$ also allows one to recover the Laplace transform of A_t at fixed $t > 0$, provided the exponential variable T is independent of the underlying Brownian motion,
$$\lambda \int_0^\infty e^{-\lambda t} \mathbb{E}[A_{T_\lambda}^{(\nu)} \mid T_\lambda = t] \, \mathrm{d}t = \mathbb{E}[A_{T_\lambda}^{(\nu)}].$$

References for part 1

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Mathematical finance in 2 minutes

- Mathematical finance concerns the dynamics of the value $S_t, 0 \leq t \leq T$ of assets, and of computing the distributional properties of $f(S_T)$, for given functions f known as *derivatives*.
- A typical method is to assume a stochastic setting with asset prices governed by

$$dS_t = rS_t\,dt + \sigma S_t\,dW_t,$$

meaning that $S(t) = S_0 \exp\left((r - \sigma^2/2)t + \sigma W_t\right)$.
- For insurance purposes certain types of derivatives have wide use in the industry, especially the class of options that give the contract holder $\max(S(T) - K, 0) = (S(T) - K)^+$ at the maturity date T.

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- For insurance purposes certain types of derivatives have wide use in the industry, especially the class of options that give the contract holder $\max(S(T) - K, 0) = (S(T) - K)^+$ at the maturity date T.
- Financial arguments allow one to conclude that there is a fair price at time $t < T$ for the option

$$C_{t,T}(K) = e^{-r(T-t)}\mathbb{E}[(S_T - K)^+ | \mathcal{F}_t].$$

Problem 2: Asian options

Asian options are also known as Average-Value (AV) options, and for this class of options the payoff function replaces the value $S(T)$ with an average value

$$\mathcal{A}(T) = \frac{1}{T - t_0} \int_{t_0}^{T} S_u \, \mathrm{d}u,$$

where $t_0 < T$ is the start time of the average value. The corresponding fair price at time $t \in [t_0, T]$,

$$C_{t,T}(K') = \frac{e^{-r(T-t)}}{T - t_0} \mathbb{E}[(A_T - K')^+ | \mathcal{F}_t], \quad A_T = \int_{t_0}^{T} S_u \, \mathrm{d}u$$

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- efficiency of modern hardware
- ...but also due to gaps in the knowledge of the distribution of $A_t^{(\nu)}$.

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Assuming $t_0 < t < T$, it is possible to arrange

$$A_T = \int_{t_0}^{T} S_u \,\mathrm{d}u = \int_{t_0}^{t} S_u \,\mathrm{d}u + S(t)\int_0^{T-t} e^{(r-\sigma^2/2)u+\sigma B_u}\,\mathrm{d}u,$$

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where $B_s = W_{t+s} - W_t$. By further factoring out $S(t)$ we obtain a formula in terms of the remaining randomness at time t,

$$C_{t,T}(K'') = \frac{S(t)e^{-r(T-t)}}{T-t_0}\mathbb{E}\left[\left(\int_0^{T-t} e^{(r-\sigma^2/2)s+\sigma B_s}\,\mathrm{d}s - K''\right)^+\right],$$

where we now have $K'' = \frac{1}{S_t}(K(T-t_0) - \int_{t_0}^{t} S_u \,\mathrm{d}u)$.

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where we now have $K'' = \frac{1}{S_t}(K(T-t_0) - \int_{t_0}^{t} S_u \,\mathrm{d}u)$.
We note that if $K'' \leq 0$ the formula has the simplification

$$\mathbb{E}\left[\left(\int_0^{T-t} e^{(r-\sigma^2/2)s+\sigma B_s}\,\mathrm{d}s - K''\right)^+\right] = \mathbb{E}[A_{T-t}^{(\nu)}] - K'', \quad \nu = r - \sigma^2/2.$$

To summarize: assuming the deterministic quantity $\int_0^{t_0} S_u \,\mathrm{d}u \geq K(T - t_0)$, we obtain the formula for the Asian option

$$C_{t,T}(K) = S_t \left(\frac{1 - e^{-r(T-t)}}{r(T - t_0)} \right) - e^{-(T-t)} \left(K - \frac{1}{T - t_0} \int_{t_0}^{t} S_u \,\mathrm{d}u \right).$$

Furthermore, we see that knowledge about the distribution of A_t^{ν} is of high interest in financial applications.

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Furthermore, we see that knowledge about the distribution of A_t^{ν} is of high interest in financial applications.

- Further knowledge of the moments $\mathbb{E}[(A_t^{(\nu)})^m]$ would be useful.
- Knowledge of the distribution of other quantities such as
  - $A_t^{(\nu)} + x$,
  - $(A_t^{(\nu)} + x)^+$,
  - $\int_{t_0}^{T} \exp(-X_s) \,\mathrm{d}s$, with X_t in the class of Lévy processes,

  would be highly useful.

Much is known about the distribution of said quantities due to the research of M. Yor, D. Dufresne, and others.

We now let X be a Lévy process and review some results about the distribution of

$$Z_t = \int_0^t e^{-X_s} \, \mathrm{d}s.$$

Theorem (Salminen and Vostrikova)

Assume X_t is a Lévy process satisfying $\mathbb{E}[e^{-\lambda X_t}] = e^{-t\phi(\lambda)} < \infty$, for all $\lambda \geq 0, t \geq 0$. If further, $\phi(i) \neq \phi(j)$ for all $0 \leq i < j \leq n$, then for $n \in \mathbb{N}$, we have the formula

$$\mathbb{E}[Z_t^n] = n! \sum_{k=0}^{n-1} \frac{e^{-t\phi(k)} - e^{-t\phi(n)}}{\prod_{i=0, i\neq k}^{n} (\phi(i) - \phi(k))}.$$

The proof relies on the independence of the increments on X and can be carried out by induction.

Some further results when $t = \infty$

In some cases the functional Z_∞ is highly tractable for the same kind of analysis.

Corollary (Salminen and Vostrikova)

Let X be a Lévy process with Laplace exponent ϕ. Define $N := \min\{n \in \mathbb{N} : \phi(n) \leq 0\}$. Then,

$$\mathbb{E}[Z_\infty^n] = \begin{cases} \frac{n!}{\prod_{k=1}^n \phi(k)}, & \text{if } n < N, \\ +\infty, & \text{if } n \geq N. \end{cases}$$

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In cases that are not well-behaved for positive integer moments one can sometimes derive information from the negative moments.

Proposition (Bertoin and Yor)

Assume (1) $\mathbb{E}[e^{\lambda X_t}] = e^{t\varphi(\lambda)} < \infty, \forall t, \lambda \geq 0$, and (2) $m := \mathbb{E}[X_1] = \varphi'(0+) > 0$. Then, for all $k \in \mathbb{N}$ and $t \in (0, +\infty]$ we have $\mathbb{E}[Z_t^{-k}] < \infty$. The formula

$$\mathbb{E}[Z_\infty^{-k}] = m \prod_{i=1}^{k-1} \frac{\varphi(i)}{i}$$

holds. Moreover, if X_t lacks positive jumps, then Carleman's Criterion is satisfied and the distribution of Z_∞ is determined by its negative integer moments.

Thank you for your interest!

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