Sparse domination and weighted norm inequalities

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Singular integrals

Calderon-Zygmund operator

We say that T is a Calderon-Zygmund operator (CZO) if it is a bounded linear operator on $L^2(\mathbb{R}^d)$ and it has the representation

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)\,\mathrm{d}y, \quad x \notin \operatorname{supp} f$$

and the kernel K satisfies

- $|K(x,y)| \leq \frac{c_K}{|x-y|^d}$, $x \neq y$,

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and the kernel K satisfies

- $|K(x,y)| \leq \frac{c_K}{|x-y|^d}$, $x \neq y$,
- $|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq \omega\left(\frac{|x-x'|}{|x-y|}\right)\frac{1}{|x-y|^d}$, $|x-y| > 2|x-x'|$ for some increasing subadditve function $\omega : [0,\infty[\to [0,\infty[$ with $\omega(0) = 0$.

One of the most fundamental example is the Hilbert transform H defined by

$$Hf(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \,\mathrm{d}y.$$

- Here the kernel is $K(x,y) = \frac{1}{\pi}\frac{1}{x-y}$ and $c_K = 1$, $\omega(t) = 4t$.
- The L^2 boundedness follows from $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi)$.

Weighted norm inequalities

We say that an operator T is a bounded operator in $L^p(w)$ if

$$\|Tf\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.$$

Coifman, Fefferman, 1974

Calderon-Zymund operators are bounded in $L^p(w)$ if and only if $w \in A_p$.

Another question is the dependence of the weight on the sharp constant C.

A_2 theorem, Hytönen, 2010

A Calderon-Zygmund operator T satisfies the quantitative bound

$$\|Tf\|_{L^2(w)} \leq C[w]_{A_2}\|f\|_{L^2(w)}.$$

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Dyadic systems

For $k \in \mathbb{Z}$, let $\mathcal{D}_k$ be a family of cubes in $\mathbb{R}^d$.

Dyadic system

A family of cubes $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ is a dyadic system if it has the following properties.

1. For a fixed $\lambda > 0$, each $\mathcal{D}_k$ is a partition of $\mathbb{R}^d$ consisting of cubes of side length $2^k\lambda$.
2. If $Q, Q' \in \mathcal{D}$ then $Q \cap Q' = \{\emptyset, Q, Q'\}$.

Examples

- The standard dyadic cubes are defined by

$$\mathcal{D} := \{2^k([0,1[^d + m) : \ k \in \mathbb{Z}, m \in \mathbb{Z}\}.$$

- Start from $I_0 := [0,1[$ and for $k \in \mathbb{N}$ let

$$I_{k+1} := (I_k + (-1)^{k+1}|I_k|) \cup I_k.$$

  Then we get a dyadic system by translating the I_k and bisecting them arbitrarily many times. In $\mathbb{R}^d$ we can take Cartesian products.

Simple yet effective algorithm

- Consider some property P that the cubes in $\mathcal{D}$ may or may not satisfy.
- Let $\mathcal{Q}$ be the family of maximal cubes (w.r.t inclusion) that satisfy P.

Corollary 1 (Whitney covering lemma, 1934)

For an open set $\Omega \subset \mathbb{R}^d$, there exists a set of pairwise disjoint dyadic cubes W that satisfy

1. $\operatorname{diam}(W) \leq \operatorname{dist}(W, \Omega^{\complement}) \leq 4 \operatorname{diam}(W)$,
2. $\Omega = \bigcup_W W$.

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Proof: Choose maximal cubes that satisfy $W \subset \Omega$ and $\operatorname{diam}(W) \leq \operatorname{dist}(W, \Omega^{\complement})$. □

Corollary 2 (Calderon-Zygmund decomposition, 1952)

Assume that $f \in L^1(\mathbb{R}^d)$ and let $0 < t < \|f\|_\infty$. Then there exists a family of disjoint dyadic cubes $Q \subset \mathbb{R}^d$ such that

1. $t < \fint_Q |f| \leq 2^d t$,
2. $|f| \leq t$ a.e. in $\mathbb{R}^d \setminus \bigcup_Q Q$.

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Proof: Take a maximal collection that satisfies $t < \fint_Q |f|$. □

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Sparse domination

Sparse family of sets

A collection of sets $\mathcal{S}$ is γ-sparse if for every $S \in \mathcal{S}$ there exists subsets E_S that are pairwise disjoint and $|E_S| \geq \gamma |S|$.

Sparse operator

A sparse operator is of the form

$$T^{\mathcal{S}} f = \sum_{S \in \mathcal{S}} \mathbb{1}_S \langle f \rangle_S,$$

where $\mathcal{S}$ is a sparse family of dyadic cubes.

The sparse operator is bounded in L^p with

$$\|T^{\mathcal{S}} f\|_{L^p} \leq pp'\gamma^{-1}\|f\|_{L^p}$$

and

Cruz-Uribe–Martell–Pérez, 2010

If $w \in A_2^{\mathcal{D}}$, then the sparse operator is bounded in $L^2(w)$ with

$$\|T^{\mathcal{S}} f\|_{L^2(w)} \leq 4\gamma^{-1}[w]_{A_2}\|f\|_{L^2(w)}.$$

General sparse domination theorem

The grand maximal operator $\mathcal{M}_T$ is defined by

$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \sup_{y \in Q} |T(\mathbb{1}_{(3Q)^\complement} f)(y)|.$$

Lerner's abstract domination theorem, 2015

Let T be linear or positive sublinear. Then for every boundedly supported $f \in L^1$ and $0 < \varepsilon < 1$, there is a $(1-\varepsilon)$-sparse family $\mathcal{S}$ of dyadic cubes such that

$$|Tf| \leq \frac{c_T c_d}{\varepsilon} \sum_{S \in \mathcal{S}} \mathbb{1}_S \fint_{3S} |f|,$$

where c_d depends only on dimension and

$$c_T = \|T\|_{L^1 \to L^{1,\infty}} + \|\mathcal{M}_T\|_{L^1 \to L^{1,\infty}}.$$

The family $\{3Q : Q \in \mathcal{D}\}$ can be divided into 3^d subcollections, each of which has the same covering and nestedness properties as $\mathcal{D}$.

Corollary

Under the assumptions of Lerner's abstract domination theorem there are $3^{-d}(1-\varepsilon)$-sparse collections $\mathcal{S}_i, i = 1, \ldots, 3^d$ such that

$$|Tf| \leq \frac{c_T c_d}{\varepsilon} \sum_{i=1}^{3^d} T^{\mathcal{S}_i}|f|.$$

A_2 theorem

If T is a Calderon-Zygmund operator, then T and $\mathcal{M}_T$ map L^1 boundedly to $L^{1,\infty}$.

Thus we get

$$\|Tf\|_{L^2(w)} \leq c_T c_d [w]_{A_2} \|f\|_{L^2(w)}.$$

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