Quasiregularly elliptic closed Riemannian manifolds

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Closed Riemannian manifolds

A closed Riemannian n-manifold, $n \in \mathbb{N}_+$, is a compact topological n-manifold M equipped with a maximal C^∞-atlas and a Riemannian tensor.

Examples of closed Riemannian manifolds

- Spheres $\mathbb{S}^n$, real projective spaces $\mathbb{R}P^n$, and complex projective spaces $\mathbb{C}P^n$.
- Cartesian products and connected sums of closed Riemannian manifolds.

The connected sum of two topological n-manifolds M and N is the topological n-manifold $M \# N$ which is obtained by removing an n-dimensional ball from both M and N and then gluing along the n-dimensional boundary spheres.

Classification of closed Riemannian manifolds in dimensions one and two

Let M be a closed and connected Riemannian 1-manifold. Then M is diffeomorphic to $\mathbb{S}^1$.

Let M be a closed and connected Riemannian 2-manifold. Then M is diffeomorphic to one of the following:

- the 2-sphere $\mathbb{S}^2$,
- the connected sum $\#^g(\mathbb{S}^1 \times \mathbb{S}^1)$ of g tori for $g \geq 1$,
- the connected sum $\#^m \mathbb{R}P^2$ of m real projective planes for $m \geq 1$.

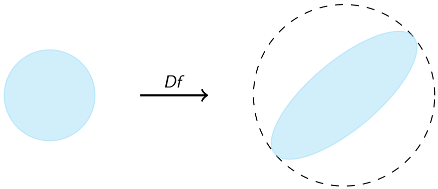
Remark: the connected sums $\#^m \mathbb{R}P^2$ are not orientable.

Quasiregularly elliptic closed Riemannian manifolds

Let M be a closed, connected, and oriented Riemannian n-manifold. Can M be covered by $\mathbb{R}^n$ in a way, which distorts the geometry of $\mathbb{R}^n$ only a uniformly bounded amount? That is, does there exist a constant $K \geq 1$ and a map $f\colon \mathbb{R}^n \to M$ for which

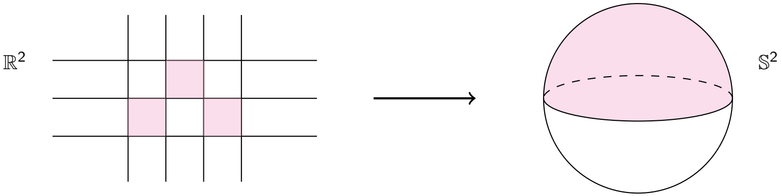
$$||Df(x)||^n \leq K \det Df(x) \quad \text{for almost every } x \in \mathbb{R}^n?$$

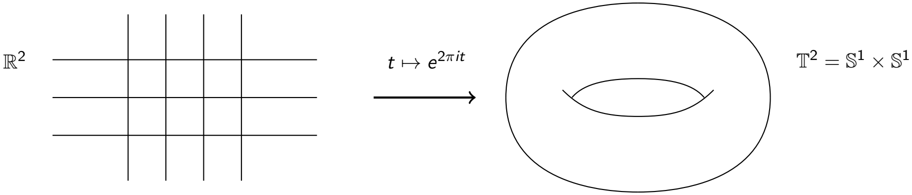
If yes, then we say that M is *quasiregularly elliptic*.



For example, the stereographic projection $\mathbb{R}^2 \to \mathbb{S}^2$ yields that $\mathbb{S}^2$ is quasiregularly elliptic.

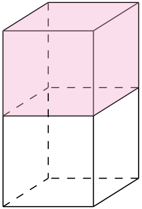
Examples for $n = 2$





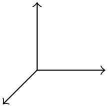
Examples for $n = 3$

$\mathbb{R}^3$



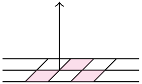
$$\longrightarrow \quad \mathbb{S}^3$$

$\mathbb{R}^3$



$$\xrightarrow{t \mapsto e^{2\pi i t}} \quad \mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$$

$\mathbb{R}^2 \times \mathbb{R} = \mathbb{R}^3$



$$\longrightarrow \quad \mathbb{S}^2 \times \mathbb{S}^1$$

Classification of quasiregularly elliptic closed Riemannian manifolds in dimensions two and three

By late 1930s

The quasiregularly elliptic closed Riemannian 2-manifolds are $\mathbb{S}^2$ and $\mathbb{T}^2$.

Early 2000s (up to conjectures late 1980s)

The quasiregularly elliptic closed Riemannian 3-manifolds are $\mathbb{S}^3$, $\mathbb{T}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, and their manifold quotients.

Let $M \in \{\mathbb{S}^3, \mathbb{T}^3, \mathbb{S}^2 \times \mathbb{S}^1\}$ and let $\sim$ be an equivalence relation on M. Then the quotient space under $\sim$ is the set $\{[x] \colon x \in M\}$ equipped with the topology, where a subset U is open if and only if $\{x \in M \colon [x] \in U\}$ is open in M.

For example, if $M = \mathbb{S}^3$ and $\sim$ is the equivalence relation identifying antipodal points, then the quotient space is $\mathbb{R}P^3$.

Examples for $n = 4$

- $\mathbb{R}^4 \to \mathbb{S}^4$
- $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4 \to \mathbb{S}^2 \times \mathbb{S}^2$
- Other (more complicated) examples?
- $\mathbb{R}^4 \to \#^2\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{R}^4 \to \#^3\mathbb{S}^2 \times \mathbb{S}^2$
- $\mathbb{R}^4 \to \#^k\mathbb{C}P^2\#^\ell\overline{\mathbb{C}P^2}$ with $k, \ell \in \{0, 1, 2, 3\}$

Constructions by Piergallini and Zuddas in 2021 give the rest of the list.

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The above list gives all the simply connected quasiregularly elliptic closed Riemannian 4-manifolds up to homeomorphism.

Thank you for your attention!