Hyperfunctions
A brief introduction

University of Helsinki

Domast student seminar
November 3, 2023

Presentation Overview

Motivation for generalized functions

Why do we need generalized functions?

- There are mathematical objects that we would like to treat as functions, but they don't quite fit into the classical definition of a function.
- The theory of generalized functions enlarges the arena, where we can do for example differential calculus and Fourier Analysis.

Distributions vs. hyperfunctions I

- The first class of generalized functions one usually encounters are the Schwartz distributions. These can be defined on C^∞-manifolds.
  - Distributions are by definition linear functionals.
- The hyperfunctions are another class of generalized functions that can be defined on C^ω(real analytic)-manifolds.
  - Hyperfunctions are boundary values of complex analytic functions is some sense.

Distributions can also be represented as boundary values of complex analytic functions. However, then the analytic functions will need to satisfy certain growth condition near the boundary.

Remove the growth conditions $\Rightarrow$ Hyperfunctions.

Distributions vs. hyperfunctions II

Why hyperfunctions?

- For example the ODE $x^2 f'(x) = f(x)$ on a domain containing 0 has non-smooth solutions, which are not distributions, but are hyperfunctions.
- The theory of hyperfunctions encompasses the distributions as part of it.

Distributions

Where do we encounter distributions

- ODE/PDE-theory
  - Solutions that are not sufficiently differentiable in classical sense.
  - The Green's functions make sense only as distributions
- Fourier analysis: Start from the test function space $\mathcal{S}(\mathbb{R}^d)$ instead of $L^1(\mathbb{R}^d)$.
- Integral operator theory: "Allows more singular kernels."
- Stochastics: There exist interesting stochastic processes, which do not have realizations on any nice function space, but rather in some space of distributions.
- Mathematical treatment of the so called Dirac calculus used by physicists in NR Quantum theory.
- Mathematical treatment of Quantum field theory: Even the free ("trivial") field theory requires distributions to make sense.

Formal introduction to distributions: An example I

What is a distribution?

A prototypical example is the Dirac delta distribution δ_x concentrated at a point $x \in \mathbb{R}^d$.

- Physicists like to call this the Dirac delta function, but unfortunately there does not exist a classical function with the desired properties for this object.
- Most notably the two properties below are in contradiction: For a measurable function $\delta : \mathbb{R}^d \to \mathbb{R}^d$: $\delta_x(\cdot) \equiv \delta(x - \cdot) = \delta(\cdot - x)$,

$$\int_{\mathbb{R}^d} f(y)\delta_x(y)dy = f(x), \ \forall f \in C(\mathbb{R}^d) \quad \text{and} \quad \delta_x(y) = 0, \ \forall y \neq x.$$

Formal introduction to distributions: An example II

There are multiple ways to make sense of δ_x more intuitively

Set $x = 0$ and $d = 1$ for simplicity.

- As the limit of sequence of C^∞ functions: "$f_n \to \delta_0$" for eg.

$$f_n(x) := \begin{cases} 0, & \text{if } |x| \geq \frac{1}{2n} \\ n, & \text{if } |x| < \frac{1}{2n} \end{cases}$$

$$f_n(x) := \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$$

$$f_n(x) := \frac{n}{\pi}\operatorname{sinc}(nx) = \frac{1}{\pi}\frac{\sin(nx)}{x}.$$

- As the derivative of the step function $\theta(x) := \mathbf{1}_{\{x\geq 0\}}$:

$$"\,\theta'(x) = \delta_0(x)"$$

Formal introduction to distributions: An example III

- As a "density" of probability measure, which has all its mass concentrated at a single point: "$d\mathbb{P}(x) = \delta_0(x)dx$"
- As a boundary values of analytic functions: Breit-Wigner formula

$$f_\epsilon(x) = \frac{i}{2\pi}\left[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}\right] \Rightarrow \text{"} \lim_{\epsilon\searrow 0} f_\epsilon = \delta_0 \text{"}$$

  Note that $z \mapsto \frac{1}{z}$, $z = x + iy$ is analytic in both the open lower and the upper half planes.

Three of these generalize to arbitrary distributions so these are the ways to characterize distributions

Most important to us is the last one since it is the contact point to the theory of hyperfunctions

Rigorous definition of distributions

Definition: The space of distributions

Let $\Omega \in \mathbb{R}^d$ be open set and $\mathcal{M}(\Omega)$ a suitable test function space. Then the space of corresponding distributions is the topological dual space $\mathcal{M}'(\Omega)$, which consists of continuous linear functionals $F\colon \mathcal{M}(\Omega) \to \mathbb{R}$ ($\mathbb{C}$).

ordinary test functions:
$\mathcal{M}(\Omega) = \mathcal{D}(\mathbb{R}^d) \equiv C_c^\infty(\mathbb{R}^d) \equiv C_0^\infty(\mathbb{R}^d)$, which are smooth and have compact support. (Could also take arbitrary $\Omega \subset \mathbb{R}^d$.)

- This yields the space of ordinary distributions $\mathcal{D}'(\mathbb{R}^d)$ ($\mathcal{D}'(\Omega)$).

Thus, rigorously we define $\delta_x[\varphi] = \varphi(x)$, $\forall \varphi \in \mathcal{D}(\mathbb{R}^d)$ (or $\forall \varphi \in \mathcal{D}(\Omega)$ with $\delta_x[\varphi] = 0$ if $x \notin \Omega$.)

The continuity of a linear map $F : \mathcal{D}(\Omega) \to \mathbb{R}$

One possible way to characterize the continuity of a linear map $F : \mathcal{D}(\Omega) \to \mathbb{R}(\mathbb{C})$ is the following:

Thm: Continuity of linear functionals on $\mathcal{D}(\Omega)$

Denote $\|\varphi\|_{K,j} = \sup_{|\alpha|\leq j} \sup_{x\in K} |(\partial^\alpha \varphi)(x)|$ for compact $K \subset \Omega$ and $\varphi \in \mathcal{D}(\Omega)$. Then $F\colon \mathcal{D}(\Omega) \to \mathbb{R}$ is continuous iff $\forall K \in \Omega$ compact, $\exists j \in \mathbb{N}_0$ and $C_K > 0$ such that

$$|\langle F, \varphi\rangle| \leq C_K \|\varphi\|_{K,j}\,.$$

Sometimes the following concept is necessary.

Definition:The order of a distribution

The order of $F \in \mathcal{D}'(\Omega)$ is the smallest integer for, which the above holds for all compact sets simultaneously. If this does not exist then the order is infinite.

Support of a distribution

Let $F \in \mathcal{D}'(\Omega)$ and $U \subset \Omega$ an open set. Then F is said to be zero on U if $F[\varphi] = 0$ for all $\varphi \in \mathcal{D}(U)$.

Definition: the support of a distribution

The support of F is the complement of the largest open set on which F is zero, that is, $\mathrm{supp}(F) = (\bigcup U)^c$, where the union is over all open $U \subset \Omega$ such that F is zero on U.

Remark: To see that this definition makes sense we would the localization/decomposition theorems of distributions.

Examples of distribution spaces I

Other possible test function spaces

- The space of smooth rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$ (also called the space of Schwartz functions). NOTE: Needs $\Omega = \mathbb{R}^d$!
  - These and all their derivatives decrease more rapidly than any inverse monomial: $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $|x|^k(\partial^\alpha \varphi)(x) \overset{|x|\to\infty}{\longrightarrow} 0$, $\forall k \in \mathbb{N}$ and for all multi-indeces α.
- The space of smooth functions $\mathcal{E}(\mathbb{R}^d) \equiv C^\infty(\mathbb{R}^d)$ (could take open $\Omega \subset \mathbb{R}^d$)

The duals

- $\mathcal{S}'(\mathbb{R}^d)$=tempered distributions
- $\mathcal{E}'(\mathbb{R}^d)$=distributions with compact support

We have the following embeddings

$$\mathcal{D}(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{E}(\mathbb{R}^d) \hookrightarrow \mathcal{E}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{D}'(\mathbb{R}^d).$$

Examples of distribution spaces II

More possible test function spaces and further embeddings

- Continuous functions with compact support: $\mathcal{M}(\Omega) = C_c(\Omega)$
  - $\mathcal{M}'(\Omega) = M_R(\Omega)$ the space of all Radon Measures with support on Ω.
    - **Remark**: One could take this as the definition of Radon measures!
- Sobolev spaces: $\mathcal{M}(\Omega) = W^{k,2}(\Omega) \equiv H^k(\Omega)$
  - $\mathcal{M}'(\Omega) = H^{-k}(\Omega)$

For our prototype distribution we have
$\delta_x \in \mathcal{D}'(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d), \mathcal{E}'(\mathbb{R}^d), M_R(\mathbb{R}^d)$ and $\delta_x \in H^{-k}(\mathbb{R}^d)$, $\forall k > \frac{d}{2}$.

- We also have the embedding $L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ given by

$$f[\varphi] := \int_\Omega f(x)\varphi(x)dx \quad \text{for } f \in L^1_{loc}(\Omega),\ \forall \varphi \in \mathcal{D}(\Omega),$$

where

$$L^1_{loc}(\Omega) := \{f \text{ measurable} | \int_\Lambda |f(x)|dx < \infty,\ \text{for all compact } \Lambda \subset \Omega\}$$

Topologies and notion of convergence I

The test function spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{E}(\Omega)$ are countably seminormed spaces. There are different equivalent families of seminorms.

- They are metrizable and even Frechet spaces (Complete with respect to the induced Frechet metric).
- They are NOT Banach spaces!

The test function spaces $\mathcal{D}(\Omega)$ and $C_c(\Omega)$ are not metrizable and their topology cannot be induced by a countable family of seminorms.

- Their topology is defined via so called strict inductive limit of Frechet spaces of similar structure on fixed compact sets $K \in \Omega$.
- They are also called LF-spaces

Topologies and notion of convergence II

Definition: Seminorm

On a Vector space X over $\mathbb{K} = \mathbb{R}, \mathbb{C}$, a mapping $p : X \to \mathbb{R}$ satisfying

1. $p(x+y) \leq p(x) + p(y)$
2. $p(\lambda x) = |\lambda| p(x)$
3. $p(0) = 0$ and $p(x) \geq 0$

for all $x, y \in X$ and $\lambda \in \mathbb{K}$ is called a seminorm on X. NOTE: $p(x) = 0 \not\Rightarrow x = 0$!

Definition:Induced Frechet metric

The mapping $d : X \times X \to \mathbb{R}$.

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(x-y)}{1 + p_n(x-y)}$$

is called the Frechet metric induced by the countable family $\mathscr{P} := \{p_n\}_{n \in \mathbb{N}}$ of seminorms p_n.

The family $\mathscr{P}$ induces the same topology as the metric d.

Topologies and notion of convergence III

Without explicitly defining the topology we can characterize the convergence in the space of ordinary test functions:

Convergence of sequences in $\mathcal{D}(\mathbb{R}^d)$ $(\mathcal{D}(\Omega))$:

A sequence $(\varphi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{D}(\mathbb{R}^d)$ $(\mathcal{D}(\Omega))$ if and only if $\exists K \in \mathbb{R}^d$ (Ω) compact and such that

- $\mathrm{supp}(\varphi_n) \in K$ for all $n \in \mathbb{N}$
- $\lim_{n\to\infty} \sup_{x\in K} |(\partial^\alpha \varphi_n)(x) - (\partial^\alpha \varphi)(x)| = 0$ for all multi-indeces $\alpha \in \mathbb{N}_0^d$, where by $\partial^\alpha f$ with $\alpha = (0, \ldots, 0)$ we mean just the function f itself.

For the spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{E}(\Omega)$, the convergence with respect to the induced Frechet metric is equivalent to convergence in the topology induced by the seminorms.

Remark: Due to continuity $\varphi_n \to \varphi$ implies $F(\varphi_n) \to F(\varphi)$ for φ in any of the above spaces and F in its dual.

Topologies and notion of convergence IV

On the distribution spaces $\mathcal{D}'(\mathbb{R}^d)$, $\mathcal{E}'(\mathbb{R}^d)$ (or with Ω) and $\mathcal{S}'(\mathbb{R}^d)$ (or more generally on the dual of TVS) one can define multiple topologies (see eg. Treves sec. 19).

- The weak-$*$ topology is probably the most commonly used.

Definition: Convergence of sequences $\mathcal{M}'(\mathbb{R}^d)$ ($\mathcal{M}'(\Omega)$)

A sequence $(F_n)_{n\in\mathbb{N}}$ converges to F in the weak-$*$ topology of $\mathcal{M}'(\mathbb{R}^d)$ iff $F_n(\varphi) \to F(\varphi)$ in $\mathbb{R}$ for all $\varphi \in \mathcal{M}(\mathbb{R}^d)$ (or similarly with Ω).

Operations on distributions I

Let us introduce the dual pairing of $\varphi \in \mathcal{M}(\Omega)$ and $F \in \mathcal{M}'(\Omega)$:
$\langle F, \varphi \rangle := F[\varphi]$.

Then we can define the following operations of distributions

1. **Multiplication by a smooth function**: For $f \in \mathcal{E}(\Omega)$ and $F \in \mathcal{D}'(\Omega)$, fF is an ordinary distribution satisfying $\langle fF, \varphi \rangle = \langle F, f\varphi \rangle$, $\forall \varphi \in \mathcal{D}(\Omega)$.
2. **Differentiation**: For $F \in \mathcal{M}'(\Omega)$ and all multi-indeces $\alpha := (a_1, a_2 \ldots, a_d) \in \mathbb{N}_0^d$, $\partial^\alpha F := \partial_d^{a_d} \ldots \partial_2^{a_2} \partial_1^{a_1} F$ is the distribution satisfying $\langle \partial^\alpha F, \varphi \rangle = (-1)^{|\alpha|} \langle F, \partial^\alpha \varphi \rangle$ for all $\varphi \in \mathcal{M}(\Omega)$.

   Remark:

   - *For $f \in C^\infty(\Omega)$ the definition coincides with the conventional differentiation by simple integration by parts.*
   - *Every distribution posesses derivatives of all orders and the usual rules of calculus apply*
   - *The differential operators D^α are linear and continuous.*

Operations on distributions II

3. **Fourier transform**: For $F \in \mathcal{S}'(\mathbb{R}^d)$, $\mathcal{F}F \in \mathcal{S}'(\mathbb{R}^d)$ is the tempered distribution satisfying $\langle \mathcal{F}F, \varphi \rangle = \langle F, \mathcal{F}\varphi \rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.
   - **Remark**: On $\mathcal{S}(\mathbb{R}^d)$ the Fourier transform is a bijective integral operator.
4. **Convolution with a test function**: Denote $\varphi_x(y) = \varphi(x-y)$ and note that $\varphi \in \mathcal{D}(\mathbb{R}^d)$ implies that the function $y \mapsto \varphi(x-y)$ is also in $\mathcal{D}(\mathbb{R}^d)$. Then the convolution of $F \in \mathcal{D}'(\mathbb{R}^d)$ with $\varphi \in \mathcal{D}(\mathbb{R}^d)$ is the smooth function

$$(F * \varphi)(x) := \langle F, \varphi_x \rangle = \langle F, \varphi(x - \cdot) \rangle.$$

We can also define the convolution and tensor product of two distributions, but we CANNOT define an "ordinary" multiplication on $\mathcal{M}'(\Omega)$ or take non-linear functions of distributions.

Now we can state the three general characterizations of distributions

The last one of these is the point of contact with Hyperfunctions.

distributions as derivatives of continuous functions

Let F_u denote the the distribution associated to the function $u \in L^1_{loc}(\Omega)$ defined by $F_u[\varphi] := \int_\Omega u(x)\varphi(x)dx$ for all $\varphi \in \mathcal{D}'(\Omega)$.

THM:Distributions as weak derivatives of continuous functions (Prop. 5.2 in [Blanchard and bruning])

Let $F \in \mathcal{D}'(\Omega)$. Then for each multi-index α, there exists a function $u_\alpha \in C(\Omega)$ such that

1. Each compact set $K \in \Omega$ intersects with the supports of only finitely many u_α
2. For all $\varphi \in \mathcal{D}(\Omega)$,

$$\langle F, \varphi \rangle = \sum_\alpha \langle \partial^\alpha F_{u_\alpha}, \varphi \rangle.$$

If the order of F is finite, then only finitely many u_α are needed for such a representation of F.

Distributions as limits of sequences of smooth functions

Definition: Regularizing sequence (Def 7.1 in [Blanchard and bruning])

A sequence $(\varphi_j)_{j\in\mathbb{N}}$ with $\varphi_j \in \mathcal{D}(\mathbb{R}^d)$, is called regularizing sequence iff

1. $\exists \varphi \in \mathcal{D}(\mathbb{R}^d)$ s.t. $\varphi_j(x) = j^d \varphi(jx)$, $\forall x \in \mathbb{R}^d$ and $j \in \mathbb{N}$.
2. $0 \leq \varphi_j(x)$ for all $x \in \mathbb{R}^d$ and $j \in \mathbb{N}$.
3. $\int_{\mathbb{R}^d} \varphi_j(x)dx = 1$ for all $j \in \mathbb{N}$.

Then we have the following theorem

THM:Approximation of distributions (Thm 7.2 in [Blanchard and bruning])

Let $T \in \mathcal{D}'(\mathbb{R}^d)$ and $(\varphi_j)_{j\in\mathbb{N}}$ be arbitrary distribution and regularizing sequence respectively. Then $(T_j)_{j\in\mathbb{N}}$ with $T * \varphi_j \in C^\infty(\mathbb{R}^d)$ converges to T in the weak topology of $\mathcal{D}'(\mathbb{R}^d)$.

Thus, every distribution is a limit of sequence of smooth functions (Or more properly since there is no uniqueness an equivalence class of Cauchy sequences of smooth functions)

Distributions as boundary values of analytic functions I

For simplicity we work in $d = 1$ from now on!

Let u be a holomorphic function on the upper half plane $\mathbb{C}^+ := \{z = x + iy \in \mathbb{C} | y > 0\}$ and let $F_{u,y}$, $y > 0$ denote the family of distributions defined by

$$\langle F_{u,y}, \varphi \rangle := \int_\Omega u(x + iy)\varphi(x)dx$$

for all $\varphi \in \mathcal{D}(\Omega)$.

Definition: Boundary value of a holomorphic function

Let u be a holomorphic function on $\mathbb{C}^+$. Then u is said to have a boundary value $u_+ \in \mathcal{D}'(\Omega)$ iff $\lim_{y \searrow 0} \langle F_{u,y}, \varphi \rangle := \langle u_+, \varphi \rangle$ exists in $\mathbb{C}$ for every $\varphi \in \mathcal{D}(\Omega)$.

We have analogous definition for the lower half plane $\mathbb{C}^- = \{z = x + iy \in \mathbb{C} | y < 0\}$!

Distributions as boundary values of analytic functions II

Theorem:Distributions as boundary values of analytic functions(Thm 11.3 in [Blanchard and Bruning] ($a = 1$), or thm 3.19 in [Berenstein and Gay])

For every $F \in \mathcal{D}'(\Omega)$ there exists an analytic function u on $[\mathrm{supp}(F)]^c$ satisfying the following

1. For every compact $K \subset \Omega$ there exists constants $a, C > 0$ and $m \in \mathbb{N}$

$$|u(x+iy)| \leq \frac{C}{|y|^m}, \quad \forall x \in K \text{ and } |y| \in (0, a]$$

   We call u satisfying this condition a function of slow growth.

2. $$\langle F, \varphi \rangle = \lim_{\epsilon \searrow 0} \int_\Omega [u(x+i\epsilon) - u(x-i\epsilon)]\varphi(x)dx$$

   for all $\varphi \in \mathcal{D}(\Omega)$.

Then we write $F = u(\cdot + i0) - u(\cdot - i0) \equiv u_+ - u_-$.

Distributions as boundary values III

Outline of the proof:

1. Begin with $F \in \mathcal{E}'(\Omega)$
   - The Cauchy transform $\widehat{F}$ of $F \in \mathcal{E}'(\Omega)$ defined by $\widehat{F}(z) := \frac{1}{2\pi i}\langle F, \frac{1}{\cdot - z}\rangle$ for $z \in [\mathrm{supp}(F)]^c$ is an analytic function on its domain of definition and satisfies the growth conditions.
2. Show that

$$\langle F, \varphi\rangle = \lim_{\epsilon \searrow 0} \int_\Omega [\widehat{F}(x + i\epsilon) - \widehat{F}(x - i\epsilon)]\varphi(x)dx.$$

   for all $\varphi \in \mathcal{D}(\Omega)$
3. Approximate a general distribution $F \in \mathcal{D}'(\Omega)$ with distributions of compact support $F_n \in \mathcal{E}'(\Omega)$. This is done by a standard argument with partition of unity on compact exhaustion of Ω.

Notation

For the rest of the presentation we will fix the following notations.

- Let $W \subset \mathbb{C}$ be a non-empty open set, then $\mathscr{H}(W)$ denotes the space of holomorphic functions.
    - It is endowed with the topology of uniform convergence on compact sets (This is a Frechet space).
- Denote $\mathbb{C}^* := \mathbb{C} \setminus \mathbb{R} = \mathbb{C}^+ \cup \mathbb{C}^-$ and let $\Omega \subset \mathbb{R}$ be a non-empty open set.
- Then set $\tilde{\Omega} := \mathbb{C}^* \cup \Omega$. We also denote

$$\mathfrak{U}(\Omega) := \{V \in \mathbb{C} \text{ open } | V \cap \mathbb{R} = \Omega\}$$

  and call this the set of complex neighbourhoods of Ω.
  Note that Ω is closed in any $V \in \mathfrak{U}(\Omega)$.
- For $W \subset \mathbb{C}$ open we denote $W^\sigma := W \cap \mathbb{C}^\sigma$
- Lastly for any $V_1, V_2 \in \mathfrak{U}(\Omega)$ with $V_1 \subset V_2$ we denote by $\rho_{V_2}^{V_1}$ the restriction map $\mathscr{H}(V_2) \to \mathscr{H}(V_1)$.

Definition and representations of hyperfunctions I

Definition: The space of hyperfunctions

We call the complex vector space

$$\mathscr{B}(\Omega) := \mathscr{H}(\mathbb{C}^*)/\rho_{\tilde{\Omega}}^{\mathbb{C}^*}(\mathscr{H}(\tilde{\Omega}))$$

the space of hyperfunctions on Ω and a hyperfunction on Ω is by definition an element in this space.

Remark 1: Since this space is defined as a quotient space an element in this space is an equivalence class.

Remark 2: Heuristically by this definition a hyperfunction can be thought as being a holomorphic function on $\mathbb{C}^*$ modulo a holomorphic function on a larger domain $\tilde{\Omega}$ restricted to the smaller domain.

Definition and representations of hyperfunctions II

Theorem

For every $V \in \mathfrak{U}(\Omega)$ there exists a natural map

$$\mathscr{B}(\Omega) \xrightarrow{i_V} \mathscr{H}(V \setminus \Omega)/\rho_V^{V\setminus\Omega}(\mathscr{H}(V)) := \mathscr{B}_V(\Omega)$$

that assigns $F := [f] \in \mathscr{B}(\Omega)$ with $f \in \mathscr{H}(V\mathbb{C}^*)$ to $F' := [f|_{V\cap\mathbb{C}^*}] \in \mathscr{B}_V(\Omega)$ and which is an isomorphism, that is,

$$\mathscr{B}(\Omega) \cong \mathscr{B}_V(\Omega).$$

- The map is induced by the restriction mapping $\rho_{V\setminus\Omega}^{\mathbb{C}^*}$.
- The fact that i_V is well defined, linear and injective are proved by elementary considerations from the definitions.
- Proof of surjectivity uses Mittag-Lefler theorem.

Remark: For any $V \in \mathfrak{U}(\Omega)$, $i_V^{-1}([g])$ with $g \in \mathscr{H}(V \setminus \Omega)$ is hyperfunction on Ω and this representation is unique. g is called the defining function of the hyperfunction. Thus, we identify $F \equiv F'$ from above.

Theorem

Let $V_1, V_2 \in \mathfrak{U}(\Omega)$ and $g_i \in \mathscr{H}(V_i \setminus \Omega)$ be the defining function of $F_i \in \mathscr{B}(\Omega)$, $i = 1, 2$. Then $F_1 = F_2$ iff $g_1|_{V_1 \cap V_2 \cap \mathbb{C}^*} - g_2|_{V_1 \cap V_2 \cap \mathbb{C}^*}$ can be analytically continued across Ω.

In particular, If $\mathscr{B}(\Omega) \ni F := [f]$ with $f \in \mathscr{H}(\mathbb{C}^*)$ and $V \in \mathfrak{U}(\Omega)$, then $g \in \mathscr{H}(V \setminus \Omega)$ represents F iff there exist $h \in \mathscr{H}(V)$ such that $f|_{V \cap \mathbb{C}^*} = g|_{V \cap \mathbb{C}^*} + h|_{V \cap \mathbb{C}^*}$.

Remark

We can decompose $\mathscr{H}(V \setminus \Omega) \ni \varphi = \varphi^+ + \varphi^-$ with $\varphi^\sigma \in \mathscr{H}([V \setminus \Omega]^\sigma)$.

Then φ represents $0 \in \mathscr{B}(\Omega)$ iff φ^+ can be analytically continued to φ^- and vice versa. Thus, F is the zero hyperfunction iff the defining function g can be analytically continued across Ω.

Operations on hyperfunctions I

1. **Multiplication by an analytic function**:

Definition

Let $V \in \mathfrak{U}(\Omega)$, $g \in \mathscr{H}(V \setminus \Omega)$ is the defining function of $F \in \mathscr{B}(\Omega)$ and $h \in \mathscr{H}(V)$. Then $Fh = hF$ is the hyperfunction on Ω with $hg \in \mathscr{H}(V \setminus \Omega)$ as the defining function.

- For $h \in \mathscr{H}(V)$, $\mathbf{1}_{V^\sigma} h \in \mathscr{H}(V \setminus \Omega)$ and the hyperfunction associated to it is $\mathbf{1}_\Omega^\sigma h$, where $\mathbf{1}_\Omega^\sigma$ is the hyperfunction on Ω associated to $\mathbf{1}_{\mathbb{C}^\sigma}$.

  Then we also have $\mathbf{1}_\Omega^+ h = -\mathbf{1}_\Omega^- h$ since obviously $\mathbf{1}_{\mathbb{C}^+} h + \mathbf{1}_{\mathbb{C}^-} h$ is an example of the previous remark.

Operations on hyperfunctions II

- *Decomposition of hyperfunctions*:

  For every $F \in \mathscr{B}(\Omega)$ with defining function $g \in \mathscr{H}(V \setminus \Omega)$, $V \in \mathfrak{U}(\Omega)$ we can write $F = F^+ + F^-$, where F^σ has the defining function $\mathbf{1}_{V^\sigma} g$.
- *This multiplication is associative*:

  For $h_1, h_2 \in \mathscr{H}(V)$ and $F \in \mathscr{B}(\Omega)$, $(h_1 + h_2)F = h_1 F + h_2 F$

2. **Conjugation**:

Definition:Conjugate of a hyperfunction

Let $V \in \mathfrak{U}(\Omega)$ and $F \in \mathscr{B}(\Omega)$ have defining function $g \in \mathscr{H}(V \setminus \Omega)$. Then the hyperfunction denoted by $\bar{F}$ and with the defining function $z \mapsto -\overline{g(\bar{z})} \in \mathscr{H}(\bar{V} \setminus \Omega)$, where $\bar{V} := \{z \in \mathbb{C} | \bar{z} \in V\}$ is called the conjugate of F.

- $\bar{\bar{F}} = F$ for all $F \in \mathscr{B}(\Omega)$

Operations on hyperfunctions III

- For $\lambda_1, \lambda_2 \in C$ and $F_1, F_2 \in \mathscr{B}(\Omega)$ we have $\overline{\lambda_1 F_1 + \lambda_2 F_2} = \bar{\lambda}_1 \bar{F}_1 + \bar{\lambda}_2 F_2$.
- The real and imaginary parts of $F \in \mathscr{B}(\Omega)$ are respectively $\mathrm{Re}\,(F) := \frac{1}{2}(F + \bar{F})$ and $\mathrm{Im}\,(F) := 1/(2i)(F - \bar{F})$.
- $F \in \mathscr{B}(\Omega)$ is called real hyperfunction if $\bar{F} = F$.

  A hyperfunction F is real precisely, when the defining function g satisfies

$$1) g \in \mathscr{H}(\mathbb{C}^*) \quad 2) g(z) = -\overline{g(\bar{z})}$$

for all $z \in \mathbb{C}^*$.

Real analytic functions and hyperfunctions

Theorem([Berenstein and Gay 2] prop. 1.2.5,1.2.7, def. 1.2.6)

The algebra $\mathcal{A}(\Omega)$ of complex valued real analytic functions can be injectively mapped into $\mathscr{B}(\Omega)$. $\mathscr{B}(\Omega)$ can be considered as an $\mathcal{A}(\Omega)$-module.

- The image of $\mathcal{A}(\Omega)$ under this injection is called the set of holomorphic hyperfunctions and they can all be written in one the following equivalent forms

$$\mathbf{1}^{+}_{\Omega} f \quad \text{or} \quad \mathbf{1}^{-}_{\Omega} g \quad \text{or} \quad \mathbf{1}^{+}_{\Omega} f_1 + \mathbf{1}^{-}_{\Omega} f_2$$

  for some $V, V_1, V_2 \in \mathfrak{U}(\Omega)$ and $f, g \in \mathscr{H}(V)$ or $f_i \in \mathscr{H}(V_i)$ with $i = 1, 2$.
- The image of $\varphi \in \mathcal{A}(\Omega)$ under this injection has $\mathbf{1}_{\mathbb{C}^* \cap V} \Phi$ as its defining function.
- If $\varphi \in \mathcal{A}(\Omega)$ and $g \in \mathscr{H}(V \setminus \Omega)$ is the defining function of $F \in \mathscr{B}(\Omega)$ with $V \in \mathfrak{U}(\Omega)$, then the hyperfunction φF has Φg as its defining function.

Above $\Phi \in \mathscr{H}(V)$ is the extension of ϕ to $V \in \mathfrak{U}(\Omega)$.

Differentiation of hyperfunctions I

Definition of the derivatives of a hyperfunction

Let $V \in \mathfrak{U}(\Omega)$ and $g \in \mathscr{H}(V \setminus \Omega)$ be the defining function of the hyperfunction $F \in \mathscr{B}(\Omega)$. Then we define the n^{th} derivative of the hyperfunction F to be the hyperfunction $D^n T \equiv T^{(n)}$ that has $g^{(n)} := \frac{d^n g}{dz^n} \in \mathscr{H}(V \setminus \Omega)$ as its defining function.

Properties of the operators $D^n : \mathscr{B}(\Omega) \to \mathscr{B}(\Omega)$

- They are linear
- $\mathcal{A}(\Omega)$ is an invariant subspace for all $n \in \mathbb{N}$
- For $\varphi \in \mathcal{A}(\Omega)$, $D^n \varphi$ coincides with the usual derivative

Differentiation of hyperfunctions II

- For $\varphi \in \mathcal{A}(\Omega)$ and $F \in \mathscr{B}(\Omega)$ the Leibniz formula

$$D^n(\varphi F) = \sum_{j=0}^{n} \binom{n}{j} (D^j \varphi) D^{p-j} F$$

  holds.
- If $\Omega = (a, b)$ for some $a < b \in \mathbb{R}$, then $F \in \mathscr{B}(\Omega)$ is a holomorphic hyperfunction with a polynomial of degree $\leq n$ as a defining function iff $F^{(n+1)} = 0$.
- The n^{th} order primitive of $F \in \mathscr{B}(\Omega)$ is the hyperfunction G satisfying $G^{(n)} = F$ unique up to a polynomial of order $\leq n - 1$.
- On $\Omega = (a, b)$ every hyperfunction admits primitives of all orders.

Examples

- Delta function $\delta_0 \in \mathscr{B}(\{0\})$ is the hyperfunction with $z \mapsto \frac{1}{2\pi i}\frac{1}{z}$ as its defining function.
- The step function $\theta \in \mathscr{B}([0,\infty)))$ is the hyperfunction with the defining function $z \mapsto \frac{-1}{2\pi i}\log(-z)$, where $\log$ is the principal branch of the complex logarithm.

Remark 1: One can easily see $\frac{d}{dz}(\frac{-1}{2\pi i}\log(-z)) = \frac{1}{2\pi i}\frac{1}{z}$, where this makes sense. Thus $\theta' = \delta_0$ as hyperfunctions.

Remark 2: Both of the defining functions satisfy the slow growth conditions so we expect that $\theta, \delta_0 \in \mathcal{D}'(\Omega)$

- Let $F \in \mathscr{B}(\Omega)$ with $0 \in \Omega$ be a hyperfunction with defining function $z \mapsto e^{\pm\frac{1}{z}}$. Then $F \notin \mathcal{D}'(\Omega)$.

Locality of hyperfunctions I

The hyperfunctions admit a concept of support!.

First we need the concept of restriction of a hyperfunction.

Definition: Restriction mapping $\mathfrak{R}_{\Omega_1}^{\Omega_2} : \mathscr{B}(\Omega_2) \to \mathscr{B}(\Omega_1)$

Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}$ be non-empty open sets and $g \in \mathscr{H}(V_2 \setminus \Omega_2)$ with $V_2 \in \mathfrak{U}(\Omega_2)$ be the defining function of $F \in \mathscr{B}(\Omega_2)$. Then the restriction of F to Ω_1 denoted by $\mathfrak{R}_{\Omega_1}^{\Omega_2}(F)$ or $F|_{\Omega_1}$ is the hyper function in $\mathscr{B}(\Omega_1)$ with defining function $g|_{(V_1 \cap V_2)\setminus\Omega_1}$ for any $V_1 \in \mathfrak{U}(\Omega_2)$.

Remark: *$F|_{\Omega_1}$ is independent of the sets V_1, V_2 and the representative defining function g. The restriction mapping $\mathfrak{R}_{\Omega_1}^{\Omega_2}$ is linear.*

Definition

We say that $F \in \mathscr{B}(\Omega)$ is zero or a holomorphic hyperfunction on an open $U \subset \Omega$ if $F|_U$ is the zero or a holomorphic hyperfunction respectively.

Locality of hyperfunctions II

Definition:Support of a hyperfunction

The support of a hyperfunction $F \in \mathscr{B}(\Omega)$ denoted by $\mathrm{supp}_\Omega(F)$ (or if clear from context just $\mathrm{supp}(F)$) is the complement of the largest open set $U \in \Omega$ such that $F = 0$ on U.

The support of $\mathrm{supp}(F)$ has the following two properties

- $\mathrm{supp}(\sum_{i=1}^n \lambda_i F_i) \subset \bigcup_{i=1}^n \mathrm{supp}(F_i)$ for any $F_i \in \mathscr{B}(\Omega)$, $\lambda_j \in \mathbb{C}$, $i = 1, 2, \ldots, n$.
- $\mathrm{supp}_{\Omega_1}(F|_{\Omega_1}) = \mathrm{supp}_{\Omega_2}(F) \cap \Omega_1$ for $F \in \mathscr{B}(\Omega)$, $\Omega_1 \subset \Omega_2$.

Locality of hyperfunctions III

The hyperfunctions have analogues of the locality and decomposition theorems of distributions.

Theorem: Localization of hyperfunctions

Let $(\Omega_j)_{j\in J}$ be an open covering of Ω, that is, Ω_j is open for all $j \in J$ and $\Omega \subset \bigcup_{j\in J} \Omega_j$. Let $T_j \in \mathscr{B}(\Omega_j)$, $j \in J$ be a collection of hyperfunctions such that for all $i, j \in J$ with $\Omega_i \cap \Omega_j \neq \emptyset$ we have

$$T_i|_{\Omega_i\cap\Omega_j} = T_j|_{\Omega_i\cap\Omega_j}.$$

Then there exists unique $T \in \mathscr{B}(\Omega)$ such that $T|_{\Omega_j} = T_j$ for all $j \in J$.

Locality of hyperfunctions IV

Theorem: Decomposition of hyperfunctions

Let $(F_j)_{j\in\mathbb{N}}$ be a locally finite sequence of of non-empty relatively closed subset sets of Ω such that $\bigcup_{j\in\mathbb{N}} F_j = \Omega$. Let $T \in \mathscr{B}(\Omega)$. Then there exists a sequence $(T_j)_{j\in\mathbb{N}}$ with $T_j \in \mathscr{B}(\Omega)$ for all $j \in \mathbb{N}$ such that $\mathrm{supp}(T_j) \subset F_j$ for all $j \in \mathbb{N}$ and such that for every open $\Omega' \subsetneq \Omega$, there exist $n(\Omega') \equiv n$ so that

$$T|_{\Omega'} = \sum_{j=1}^{n} T_j|_{\Omega'}.$$

Locality of hyperfunctions V

However, the following theorem has no analogue in the theory of distributions

Theorem

The restriction map $\mathfrak{R}^{\mathbb{R}}_{\Omega} : \mathscr{B}(\mathbb{R}) \to \mathscr{B}(\Omega)$ *is surjective, that is, for any* $F \in$ *there exists* $F_0 \in \mathbb{R}$ *such that* $F = F_0|_{\Omega}$ *and* $\mathrm{supp}(F_0) \subset \bar{\Omega}$*.*

Outline of proof:
Pick $V = \mathbb{C} \setminus \partial\Omega$, then $V \in \mathfrak{U}(\Omega)$ and the defining function $g \in \mathscr{H}(V \setminus \Omega) = \mathscr{H}(\mathbb{C} \setminus \bar{\Omega})$ of $F \in \mathscr{H}(\Omega)$. Let $F_0 \in \mathscr{B}(\mathbb{R})$ have $g|_{\mathbb{C}\setminus\mathbb{R}}$ as its defining function. Then $F_0|_{\Omega} = F$ and $\mathrm{supp}(F_0) \subset \bar{\Omega}$.

Locality of hyperfunctions VI

We can generalize this to situation where we have $\mathfrak{R}_{\Omega_1}^{\Omega_2}$ with $\Omega_1 \subset \Omega \subsetneq \mathbb{R}$ and both open. We also have the corollaries

Corollary

1 Let Ω_i, $i = 1, 2$ be as above, then

$$\mathscr{B}(\Omega_1) \cong \{F \in \mathscr{B}(\Omega_2) | \mathrm{supp}(F) \subset \bar{\Omega}_1\}.$$

Corollary

2 Let $S \subset \Omega$ be closed subset of the open set Ω. Then

$$\mathscr{B}(S) \cong \{F \in \mathscr{B}(\Omega) | \mathrm{supp}(F) \subset S\}.$$

For latter take $\Omega_2 = \Omega \setminus S$ and $\Omega_1 = \Omega$ in the former.

Distributions and hyperfunctions

Theorem

There exists a canonical injection $i\colon \mathcal{D}'(\Omega) \to \mathscr{B}(\Omega)$*.*

- *This injection preserves the support, that is,*

$$\operatorname{supp}(F) = \operatorname{supp}(i(F)) \quad \textit{for } F \in \mathcal{D}(\Omega).$$

- *The image* $i(\mathcal{D}'(\Omega))$ *is a subspace of* $\mathscr{B}(\Omega)$*.* $F \in i(\mathcal{D}'(\Omega))$ *have defining functions that satisfy the slow growth condition in* Ω*.*

Corollary

We have the following isomorphism

$$\mathcal{D}'(\Omega) \cong \mathscr{H}_{SG,\Omega}(\mathbb{C}^*)/\rho_{\tilde{\Omega}}^{\mathbb{C}^*}(\mathscr{H}(\tilde{\Omega})),$$

where $\mathscr{H}_{SG,\Omega}(\mathbb{C}^*)$ *is the space of holomorphic functions in* $\mathbb{C}^*$ *that satisfy the slow growth condition in* Ω*.*

Analytic functionals and hyperfunctions I

Definition: Analytic functional

Let $W \in \mathbb{C}$ be open set. Then an analytic functional on $W \in \mathbb{C}$ is an element of the dual space $\mathscr{H}'(W)$

Definition: Locally analytic functional

Let $K \in \mathbb{C}$ be a compact set. Then a locally analytic functional is an element in the dual space $\mathscr{H}'(K)$.

If we have open or compact subsets of the real line embedded into the complex plane and real analytic functions on these sets we use similar terminology.

Analytic functionals and hyperfunctions II

Theorem

Let $K \subset \mathbb{R} \subset \mathbb{C}$ be a compact set and $\Omega \subset \mathbb{R} \subset \mathbb{C}$ be an open set. Then we have the following isomorphisms

$$\mathcal{A}'(K) \cong \mathscr{B}(K)$$
$$\mathcal{A}'(\Omega) \cong \mathscr{B}_c(\Omega),$$

where $\mathscr{B}_c(\Omega)$ is the space of hyperfunctions on Ω with compact support.

References I

François Tréves
Topological vector spaces, Distributions and Kernels
Elsevier, 1967

Walter Rudin
Functional analysis
McGraw-Hill, 1974

Philippe Blanchard and Erwin Brüning
Mathematical methods in physics: Distributions, Hilbert spcae operators, Variational methods, and applications in quantum physics, Second edition
Birkhäuser, 2015

Yvonne Choquet-Bruhat, Cécile DeWitt-Morette (and Margaret Dillard-Bleick)
Analysis, Manifolds and physics, Part I:Basics
Elsevier /North Holland, 1982

References II

Carlos A. Berenstein and Roger Gay
Complex variables: An introduction
Springer-Verlag, 1991

Carlos A. Berenstein and Roger Gay
Complex analysis and special topics in harmonic analysis
Springer-Verlag, 1995

Mitsuo Morimoto
An introduction to Sato's Hyperfunctions
American mathematical society, 1993
Translated from original Japanese version from 1976

Lars Hörmander
The analysis of linear partial differential operators I, second edition
Springer-Verlag, 1989

The End

Questions? Comments?