Additional proofs to accompany the paper "Parameter estimation in nonlinear AR–GARCH models" by Mika Meitz and Pentti Saikkonen. (Not to be published.)

Proof of Lemma A.1. The case $r \geq 1$ follows from Minkowski's inequality. When $0 < r < 1$, Loève's c_r–inequality (see Davidson (1994), p. 140) first applied with r and then with $1/r$ yields

$$\left(E\left|\sum_{i=1}^{k} x_i\right|^r\right)^{1/r} \leq c_r^{1/r}\left(\sum_{i=1}^{k} E\left|x_i\right|^r\right)^{1/r} \leq c_r^{1/r} c_{1/r} \sum_{i=1}^{k}\left(E\left|x_i\right|^r\right)^{1/r},$$

where $c_r^{1/r} = 1$ and $c_{1/r} = k^{1/r-1}$. Hence the result. ■

Proof of Lemma A.2. By the Borel-Cantelli lemma, the first result follows if we show that $\sum_{t=1}^{\infty} P(x_t > \delta^t) < \infty$ for some $\delta \in (0, 1)$. By assumption, $\gamma^t \|x_t\|_r \to 0$, and hence we can find a $C < \infty$ such that $\|x_t\|_r \leq C\gamma^{-t}$ for all $t \in \mathbb{N}$. Hence $E[x_t^r] \leq C^r \gamma^{-tr}$ for all $t \in \mathbb{N}$. Choose a δ such that $\gamma^{-1} < \delta < 1$. Then $(\delta\gamma)^{-r} < 1$, and

$$\sum_{t=1}^{\infty} P(x_t > \delta^t) \leq \sum_{t=1}^{\infty} \delta^{-tr} E\left[x_t^r\right] \leq C^r \sum_{t=1}^{\infty} (\delta\gamma)^{-tr} < \infty.$$

This proves the former result. When $r \geq 1$ the latter result follows from the aforementioned inequality $\|x_t\|_r \leq C\gamma^{-t}$ by using Minkowski's inequality and monotone convergence. When $r < 1$ the same conclusion is obtained by using Loeve's c_r–inequality (see Davidson (1994), p. 140) instead of Minkowski's inequality (cf. the proof of Lemma A.1). ■

Additional details for Proof of Theorem 1. We give a brief outline of the required steps. Exactly as in the aforementioned proof of Pfanzagl, it can be shown that $E\left[l_t^*(\theta)\right]$ is a lower semicontinuous function on Θ and, moreover, for every $\theta \in \Theta$ there exists an open neighborhood $B(\theta)$ of θ such that $E\left[\inf_{\theta \in B(\theta) \cap \Theta} l_t^*(\theta)\right] > l_\bullet$ whenever $E\left[l_t^*(\theta)\right] > l_\bullet$ (we note that $E\left[l_t^*(\theta)\right]$ can equal ∞, and also that the property $E\left[\inf_{\theta \in \Theta} l_t^*(\theta)\right] > -\infty$ is required here so that the monotone convergence theorem applies). Now let $l_\bullet$ be such that $E\left[l_t^*(\theta)\right] > l_\bullet$ for all $\theta \in B(\theta_0, \delta)^c$. The open sets $B(\theta)$, $\theta \in B(\theta_0, \delta)^c$, form a cover of the compact set $B(\theta_0, \delta)^c$, and hence we may choose a finite subcover, say $B(\theta_{(1)})$, $\ldots$, $B(\theta_{(k)})$. Because $E\left[\inf_{\theta \in \Theta} l_t^*(\theta)\right] > -\infty$, the ergodic theorem yields

$$\liminf_{T\to\infty} \inf_{\theta \in B(\theta_{(i)}) \cap \Theta} L_T^*(\theta) \geq \lim_{T\to\infty} T^{-1} \sum_{t=1}^{T} \inf_{\theta \in B(\theta_{(i)}) \cap \Theta} l_t^*(\theta) = E\left[\inf_{\theta \in B(\theta_{(i)}) \cap \Theta} l_t^*(\theta)\right] \quad \text{a.s.,} \tag{41}$$

$i = 1, \ldots, k$, even when the expected value in (41) equals $+\infty$ (cf. Billingsley (1995), pp. 284 and 495, and Francq and Zakoïan (2004), p. 617). Making use of the inequality $\inf_{\theta \in B(\theta_0,\delta)^c} L_T^*(\theta) \geq \min_{i=1,\ldots,k} \inf_{\theta \in B(\theta_{(i)}) \cap \Theta} L_T^*(\theta)$ and (41) we obtain

$$\liminf_{T\to\infty} \inf_{\theta \in B(\theta_0,\delta)^c} L_T^*(\theta) \geq \liminf_{T\to\infty} \min_{i=1,\ldots,k} \inf_{\theta \in B(\theta_{(i)}) \cap \Theta} L_T^*(\theta) \geq \min_{i=1,\ldots,k} E\left[\inf_{\theta \in B(\theta_{(i)}) \cap \Theta} l_t^*(\theta)\right] > l_\bullet \quad \text{a.s.}$$

Because $l_\bullet$ is arbitrary, we obtain the desired result. ■

Proof of Lemma C.1. To prove part (i), first note that $\left\|\sup_{\theta\in\Theta_0} h_t^*\right\|_r < \infty$ by Proposition 1 and that u_t is L_{2r}–dominated in Θ_0, as seen in the proof of the same proposition. Thus, Assumption N3(ii) and Lemma A.1 imply that $g_{\theta,t}^*$, $g_{u,t}^*$, $g_{\theta\theta,t}^*$, $g_{uu,t}^*$, $g_{u\theta,t}^*$, and $g_{\theta u,t}^*$ are L_r–dominated in Θ_0. The Lipschitz conditions of Assumptions C2(ii) and N3(iii) ensure that $g_{h,t}^*$, $g_{\theta h,t}^*$, $g_{\theta h,t}$, $g_{uh,t}^*$, $g_{uh,t}$, and $g_{hh,t}^*$ are bounded by a finite constant uniformly over Θ_0. Moreover, Assumptions DGP and N3(i) ensure that $f_{\theta,t}$ and $f_{\theta\theta,t}$ are L_{2r}–dominated in Θ_0 (cf. the beginning of the proof of Proposition 1). The result now follows from Lemma A.1, the Cauchy-Schwartz inequality, and the norm inequality (for simplicity, the same order, $r/2$, is used for the first two terms). In (ii), the boundedness of the absolute differences follows directly from the Lipschitz conditions of Assumption N3(iii) (again, for simplicity, the same upper bound is used for all the absolute differences). As was noted above, $f_{\theta,t}$ and $f_{\theta\theta,t}$ are L_{2r}–dominated in Θ_0, and hence C_{t-1} is L_r–dominated in Θ_0 by Lemma A.1. The results in (iii) follow from the Lipschitz conditions of Assumptions C2(ii) and N3(iii). ■

Proof of Proposition 2. To prove part (a), we first apply Lemma A.3. Set $z_t(\theta) = h_{\theta,t}(\theta)$ and $v_{t-1}^*(\theta) = (\alpha_{\theta,t}^*, \beta_t^*)$. For all $v \in \mathbb{R}^{m+l+1}$, $z \in \mathbb{R}^{m+l}$, and $\theta \in \Theta_0$, define the function G as $G(v, z; \theta) = (v_1, \ldots, v_{m+l}) + v_{m+l+1} z$, where the subscript denotes a particular coordinate of the vector v. Thus $z_t(\theta) = h_{\theta,t}(\theta)$ satisfies the difference equation $z_t(\theta) = G\left(v_{t-1}^*(\theta), z_{t-1}(\theta); \theta\right)$. Condition G, the continuity of $v_t^*(\cdot)$, and the moment condition $E[\sup_{\theta\in\Theta_0} \psi\left(|v_t^*(\theta)|\right)^{r/2}] < \infty$ hold with $\psi(x) = x$ due to Assumption N2 and Lemma C.1. The results of part (a), except for the last one concerning differentiability, now follow from Lemma A.3 (note that the solution $h_{\theta,t}^*(\theta)$ is understood to be initialized from $h_{\theta,0}^*(\theta)$ having this stationary distribution).

The a.s. continuous differentiability of $h_t^*(\theta)$ and the relation $\partial h_t^*(\theta)/\partial\theta = h_{\theta,t}^*(\theta)$ a.s. can be proved in a similar manner as in Straumann and Mikosch (2006, pp. 2483–2484). To this end, let $x \in \mathbb{C}(\Theta, \mathbb{R}_+)$ be twice continuously differentiable on Θ_0 and define the sequence $\tilde{h}_n(\theta)$, $n \geq 0$, with $\tilde{h}_0(\theta) = x(\theta)$ and $\tilde{h}_n(\theta) = h_{n,n-1}(\theta)$, $n \geq 1$, where $h_{t,s} = (g_t \circ \cdots \circ g_{t-s})(x)$, $s \geq 0$, with $[g_t(x)](\theta) = g(u_{t-1}(\theta), x(\theta); \theta)$ (cf. proof of Proposition 1). Thus $\tilde{h}_n(\theta)$, $n \geq 0$, is a random sequence in $\mathbb{C}(\Theta, \mathbb{R}_+)$ with elements twice continuously differentiable on Θ_0 with probability one (the latter fact follows from Assumption N2). Moreover, $\tilde{h}_n(\theta)$ and $\tilde{h}_{\theta,n}(\theta) = \partial\tilde{h}_n(\theta)/\partial\theta$ are solutions to the difference equations (6) and (9). Hence, by part (b) of this proposition (the proof of which does not rely on the subresult currently being proven), for some $\gamma > 1$,

$$\gamma^n \sup_{\theta\in\Theta_0} |h_{\theta,n}^*(\theta) - \tilde{h}_{\theta,n}(\theta)| \to 0 \quad \text{in } L_{r/4} - \text{norm as } n \to \infty. \tag{42}$$

On the other hand, note that for any fixed $n \geq 1$, $(\partial h_{t,n-1}(\theta)/\partial\theta, h_{\theta,t}^*(\theta))$ is a stationary process. Therefore, $(\partial h_{t,n-1}(\theta)/\partial\theta, h_{\theta,t}^*(\theta))$ and $(\partial h_{n,n-1}(\theta)/\partial\theta, h_{\theta,n}^*(\theta))$ are identically distributed. In the latter, $\partial h_{n,n-1}(\theta)/\partial\theta = \tilde{h}_{\theta,n}(\theta)$, and hence, making use of (42), it also holds that $\gamma^n \sup_{\theta\in\Theta_0}|h^*_{\theta,t}(\theta) - \partial h_{t,n-1}(\theta)/\partial\theta| \to 0$ in $L_{r/4}$–norm as $n \to \infty$. By Lemma A.2, $\sup_{\theta\in\Theta_0}|h^*_{\theta,t}(\theta) - \partial h_{t,n-1}(\theta)/\partial\theta| \to 0$ a.s. as $n \to \infty$. To conclude, we have shown that $h_{t,n-1}(\theta)$ converges to $h^*_t(\theta)$ a.s. as $n \to \infty$ for each $\theta \in \Theta_0$ (see the proof of Proposition 1) and that $\partial h_{t,n-1}(\theta)/\partial\theta$ converges uniformly to $h^*_{\theta,t}(\theta)$ a.s. as $n \to \infty$. Now, by Lang (1993, Theorem XIII.9.1) and the continuity of $h^*_{\theta,t}(\theta)$ (obtained from Lemma A.3), $h^*_t(\theta)$ is a.s. continuously differentiable on Θ_0 and $\partial h^*_t(\theta)/\partial\theta = h^*_{\theta,t}(\theta)$ a.s.

To prove part (b), note that by the definitions, Lemma C.1, and denoting $a_t = C_t(1+|h^*_{\theta,t}|)$,

$$|h^*_{\theta,t}-h_{\theta,t}| \leq |\alpha^*_{\theta,t}-\alpha_{\theta,t}|+|\beta^*_t-\beta_t||h^*_{\theta,t-1}|+|\beta_t||h^*_{\theta,t-1}-h_{\theta,t-1}| \leq a_{t-1}|h^*_{t-1}-h_{t-1}|+\kappa|h^*_{\theta,t-1}-h_{\theta,t-1}|.$$

Repeated substitution yields $|h^*_{\theta,t}-h_{\theta,t}| \leq \sum_{j=0}^{t-1}\kappa^{t-1-j}a_j|h^*_j-h_j|+\kappa^t|h^*_{\theta,0}-h_{\theta,0}|$, where $h_{\theta,0}=0$. Using Lemma A.1 and Hölder's inequality we obtain

$$\Delta^{-1}_{r/4,t+1}\left\|\sup_{\theta\in\Theta_0}\left|h^*_{\theta,t}-h_{\theta,t}\right|\right\|_{r/4} \leq \sum_{j=0}^{t-1}\kappa^{t-1-j}\left\|\sup_{\theta\in\Theta_0}a_j\right\|_{r/3}\left\|\sup_{\theta\in\Theta_0}\left|h^*_j-h_j\right|\right\|_r+\kappa^t\left\|\sup_{\theta\in\Theta_0}\left|h^*_{\theta,0}\right|\right\|_{r/4}.$$

In the former term on the majorant side, $\left\|\sup_{\theta\in\Theta_0}a_j\right\|_{r/3}$ is bounded by a finite constant by Hölder's inequality, part (a), and Lemma C.1, whereas $\left\|\sup_{\theta\in\Theta_0}|h^*_j-h_j|\right\|_r \leq C\kappa^j$ by (21). Thus the former term is bounded by $C't\kappa^{t-1}$ for some finite C'. In the latter term, the norm is finite by part (a). Therefore, for some finite C'',

$$\left\|\sup_{\theta\in\Theta_0}|h^*_{\theta,t}-h_{\theta,t}|\right\|_{r/4} \leq C''\max\{t,t^{4/r}\}\kappa^{t-1},$$

from which the stated result follows. ■

Proof of Proposition 3. To prove part (a), we first apply Lemma A.3. Set $z_t(\theta) = \operatorname{vec}(h_{\theta\theta,t})$ and $v^*_{t-1}(\theta) = (\operatorname{vec}(\alpha^*_{\theta\theta,t}),\beta^*_t,\gamma^*_{\theta,t},\delta^*_t,h^*_{\theta,t})$, where $\operatorname{vec}(\cdot)$ signifies the usual columnwise vectorization of a matrix. For all $v \in \mathbb{R}^{(m+l+1)^2+1}$, $z \in \mathbb{R}^{(m+l)^2}$, and $\theta \in \Theta_0$, define the function G as $G(v,z;\theta) = v_1 + v_2 z + \operatorname{vec}(v_3 v_5') + \operatorname{vec}(v_5 v_3') + v_4\operatorname{vec}(v_5 v_5')$, where $v = (v_1,v_2,v_3,v_4,v_5)$ is partitioned conformably with the partition of $v^*_{t-1}(\theta)$ above. Thus $z_t(\theta) = \operatorname{vec}(h_{\theta\theta,t})$ satisfies the difference equation $z_t(\theta) = G\left(v^*_{t-1}(\theta), z_{t-1}(\theta);\theta\right)$. Condition G as well as the moment condition $E[\sup_{\theta\in\Theta_0}\psi(|v^*_t(\theta)|)^{r/4}] < \infty$ hold with $\psi(x) = \bar{\varkappa}x^2+\bar{\varpi}$ $(0<\bar{\varkappa},\bar{\varpi}<\infty)$ due to the Cauchy-Schwarz inequality, Proposition 2, and Lemmas A.1 and C.1. The results of part (a), except for the last one concerning differentiability, now follow from Lemma A.3 (with $h^*_{\theta\theta,t}(\theta)$ being initialized from $h^*_{\theta\theta,0}(\theta)$ having this stationary distribution). Finally, the proof of a.s. differentiability and of the relation $\partial^2 h^*_t(\theta)/\partial\theta\partial\theta' = h^*_{\theta\theta,t}(\theta)$ a.s. is analogous to that in Proposition 2, cf. Straumann and Mikosch (2006, pp. 2485–2486). We omit the details for brevity, and only note that the result of part (b) is needed to prove this.

To prove part (b), note that by the definitions

$$\begin{aligned}\left|h^*_{\theta\theta,t}-h_{\theta\theta,t}\right| &\leq \left|\alpha^*_{\theta\theta,t}-\alpha_{\theta\theta,t}\right|+\left|\beta^*_t h^*_{\theta\theta,t-1}-\beta_t h_{\theta\theta,t-1}\right|+\left|\gamma^*_{\theta,t}h^{*\prime}_{\theta,t-1}-\gamma_{\theta,t}h^{\prime}_{\theta,t-1}\right| \\ &\quad +\left|h^*_{\theta,t-1}\gamma^{*\prime}_{\theta,t}-h_{\theta,t-1}\gamma^{\prime}_{\theta,t}\right|+\left|\delta^*_t h^*_{\theta,t-1}h^{*\prime}_{\theta,t-1}-\delta_t h_{\theta,t-1}h^{\prime}_{\theta,t-1}\right|. \qquad (43)\end{aligned}$$

The second, third (which equals the fourth), and fifth term on the majorant side of (43) are bounded from above by $|\beta^*_t-\beta_t||h^*_{\theta\theta,t-1}|+|\beta_t||h^*_{\theta\theta,t-1}-h_{\theta\theta,t-1}|$, $|\gamma^*_{\theta,t}-\gamma_{\theta,t}||h^*_{\theta,t-1}|+|\gamma_{\theta,t}||h^*_{\theta,t-1}-h_{\theta,t-1}|$, and $|\delta^*_t-\delta_t||h^*_{\theta,t-1}h^{*\prime}_{\theta,t-1}|+|\delta_t||h^*_{\theta,t-1}h^{*\prime}_{\theta,t-1}-h_{\theta,t-1}h^{\prime}_{\theta,t-1}|$, respectively. In the last of these upper bounds, $|h^*_{\theta,t-1}h^{*\prime}_{\theta,t-1}-h_{\theta,t-1}h^{\prime}_{\theta,t-1}|\leq 2|h^*_{\theta,t-1}||h^*_{\theta,t-1}-h_{\theta,t-1}|+|h^*_{\theta,t-1}-h_{\theta,t-1}|^2$. Using these inequalities and Lemma C.1 we obtain the following inequalities for the four distinct terms on the majorant side of (43):

$$\begin{aligned}\left|\alpha^*_{\theta\theta,t}-\alpha_{\theta\theta,t}\right| &\leq C_{t-1}\left|h^*_{t-1}-h_{t-1}\right|, \\ \left|\beta^*_t h^*_{\theta\theta,t-1}-\beta_t h_{\theta\theta,t-1}\right| &\leq C_{t-1}\left|h^*_{t-1}-h_{t-1}\right|\left|h^*_{\theta\theta,t-1}\right|+\kappa\left|h^*_{\theta\theta,t-1}-h_{\theta\theta,t-1}\right|, \\ \left|\gamma^*_{\theta,t}h^{*\prime}_{\theta,t-1}-\gamma_{\theta,t}h^{\prime}_{\theta,t-1}\right| &\leq C_{t-1}\left|h^*_{t-1}-h_{t-1}\right|\left|h^*_{\theta,t-1}\right|+\left|\gamma_{\theta,t}\right|\left|h^*_{\theta,t-1}-h_{\theta,t-1}\right|, \\ \left|\delta^*_t h^*_{\theta,t-1}h^{*\prime}_{\theta,t-1}-\delta_t h_{\theta,t-1}h^{\prime}_{\theta,t-1}\right| &\leq C_{t-1}\left|h^*_{t-1}-h_{t-1}\right|\left|h^*_{\theta,t-1}h^{*\prime}_{\theta,t-1}\right| \\ &\quad +\kappa'(2\left|h^*_{\theta,t-1}\right|\left|h^*_{\theta,t-1}-h_{\theta,t-1}\right|+\left|h^*_{\theta,t-1}-h_{\theta,t-1}\right|^2).\end{aligned}$$

Denoting $b_{t-1}=C_{t-1}(1+2|h^*_{\theta,t-1}|+|h^*_{\theta,t-1}|^2+|h^*_{\theta\theta,t-1}|)$ and $c_{t-1}=2|\gamma_{\theta,t}|+2\kappa'|h^*_{\theta,t-1}|$ we obtain

$$\left|h^*_{\theta\theta,t}-h_{\theta\theta,t}\right|\leq b_{t-1}\left|h^*_{t-1}-h_{t-1}\right|+c_{t-1}\left|h^*_{\theta,t-1}-h_{\theta,t-1}\right|+\kappa'\left|h^*_{\theta,t-1}-h_{\theta,t-1}\right|^2+\kappa\left|h^*_{\theta\theta,t-1}-h_{\theta\theta,t-1}\right|.$$

By repeated substitution

$$\left|h^*_{\theta\theta,t}-h_{\theta\theta,t}\right|\leq\sum_{j=0}^{t-1}\kappa^{t-1-j}\left(b_j\left|h^*_j-h_j\right|+c_j\left|h^*_{\theta,j}-h_{\theta,j}\right|+\kappa'\left|h^*_{\theta,j}-h_{\theta,j}\right|^2\right)+\kappa^t\left|h^*_{\theta\theta,0}-h_{\theta\theta,0}\right|,$$

where $h_{\theta\theta,0}=0$. Using Lemma A.1, Hölder's inequality, and the norm inequality

$$\begin{aligned}&\Delta^{-1}_{r/8,3t+1}\left\|\sup_{\theta\in\Theta_0}\left|h^*_{\theta\theta,t}-h_{\theta\theta,t}\right|\right\|_{r/8} \\ \leq\ &\sum_{j=0}^{t-1}\kappa^{t-1-j}\left\|\sup_{\theta\in\Theta_0}b_j\right\|_{r/5}\left\|\sup_{\theta\in\Theta_0}\left|h^*_j-h_j\right|\right\|_r+\sum_{j=0}^{t-1}\kappa^{t-1-j}\left\|\sup_{\theta\in\Theta_0}c_j\right\|_{r/2}\left\|\sup_{\theta\in\Theta_0}\left|h^*_{\theta,j}-h_{\theta,j}\right|\right\|_{r/4} \\ &+\sum_{j=0}^{t-1}\kappa^{t-1-j}\kappa'\left\|\sup_{\theta\in\Theta_0}\left|h^*_{\theta,j}-h_{\theta,j}\right|^2\right\|_{r/8}+\kappa^t\left\|\sup_{\theta\in\Theta_0}\left|h^*_{\theta\theta,0}\right|\right\|_{r/4}.\end{aligned}$$

By arguments already used, the terms $\|\sup_{\theta\in\Theta_0}b_j\|_{r/5}$, $\|\sup_{\theta\in\Theta_0}c_j\|_{r/2}$, and $\|\sup_{\theta\in\Theta_0}|h^*_{\theta\theta,0}|\|_{r/4}$ are bounded by a finite constant. Furthermore, by (21) and (26), the terms $\|\sup_{\theta\in\Theta_0}|h^*_j-h_j|\|_r$ and $\|\sup_{\theta\in\Theta_0}|h^*_{\theta,j}-h_{\theta,j}|\|_{r/4}$ are bounded from above by $C'\kappa^j$ and $C'\max\{j,j^{4/r}\}\kappa^j$, respectively, for some finite C'. Therefore, for some finite C'',

$$\left\|\sup_{\theta\in\Theta_0}\left|h^*_{\theta\theta,t}-h_{\theta\theta,t}\right|\right\|_{r/8}\leq C''\Delta_{r/8,3t+1}\left(t\kappa^{t-1}+t\max\{t,t^{4/r}\}\kappa^{t-1}+\kappa^t\right),$$

from which the result follows. ■

Proof of Lemma D.1, remaining details. As noted in the discussion following Assumption N4, the moment condition $E[\varepsilon_t^8] < \infty$ can be weakened to $E[\varepsilon_t^4] < \infty$ if the i.i.d. assumption is made. The reason is that then the terms involving ε_t in the expression of $E\left[l_{\theta,t}^*(\theta_0)l_{\theta,t}^{*\prime}(\theta_0)\right]$ can be factored out. On the other hand, when the errors are allowed to be dependent, requiring $E[|\varepsilon_t|^{4+\delta}] < \infty$ for some $\delta > 0$ suffices in the linear pure GARCH case because then, unlike in our present case, the term $|h_{\theta,t}^*(\theta_0)|/\sigma_t^2$ possesses moments of any order; see Escanciano (2009). Alternatively, if $E[\varepsilon_t^4] < \infty$ and $E[\varepsilon_t^4 \mid \mathcal{F}_{t-1}] \leq K < \infty$ a.s. are assumed, finiteness of $E\left[l_{\theta,t}^*(\theta_0)l_{\theta,t}^{*\prime}(\theta_0)\right]$ also follows as in Lee and Hansen (1994, p. 49) (this can be justified by using the law of iterated expectations and other common properties of conditional expectations which hold true for possibly non-integrable random variables, see Loève (1978, Sections 27–28)). ■

Proof of Lemma D.3. Consider the matrix $\mathcal{I}(\theta_0)$. For an arbitrary $x = (x_\mu, x_\lambda) \in \mathbb{R}^m \times \mathbb{R}^l$, suppose $x'\mathcal{I}(\theta_0)\,x = E\left[\left(x'l_{\theta,t}^*(\theta_0)\right)^2\right] = 0$. Then, by (29), we must have

$$x'l_{\theta,t}^*(\theta_0) = 2\varepsilon_t x'\frac{f_{\theta,t}(\theta_0)}{\sigma_t} + \left(\varepsilon_t^2 - 1\right)x'\frac{h_{\theta,t}^*(\theta_0)}{\sigma_t^2} = 0 \quad \text{a.s.}$$

Now the proof proceeds as in Francq and Zakoïan (2004), their derivation between equations (4.52) and (4.53), but with the i.i.d. assumption used therein replaced by Assumption E (this means that instead of ordinary expectations we use expectations conditional on $\mathcal{F}_{t-1}$ so that the third and fourth moments of the errors that appear in Francq and Zakoïan (2004) will be replaced by their conditional counterparts $E[\varepsilon_t^3 \mid \mathcal{F}_{t-1}]$ and $E[\varepsilon_t^4 \mid \mathcal{F}_{t-1}]$). Using Assumption N5(i) instead of its unconditional counterpart we can therefore conclude that, almost surely, $x_\mu' f_{\mu,t}(\theta_0) = 0$ and $x'h_{\theta,t}^*(\theta_0) = 0$. By Assumption N5(ii), $x_\mu = 0$, and hence $x_\lambda' h_{\lambda,t}^*(\theta_0) = 0$ a.s. By equation (11) and the definitions preceding it in Section 4,

$$\begin{aligned} h_{\lambda,t}^*(\theta_0) &= \alpha_{\lambda,t}^*(\theta_0) + \beta_t^*(\theta_0)\,h_{\lambda,t-1}^*(\theta_0) \\ &= \partial g(u_{0,t-1}, \sigma_{t-1}^2; \theta_0)/\partial\lambda + \partial g(u_{0,t-1}, \sigma_{t-1}^2; \theta_0)/\partial h \cdot h_{\lambda,t-1}^*(\theta_0) \quad \text{a.s.} \end{aligned}$$

By stationarity, also $x_\lambda' h_{\lambda,t-1}^*(\theta_0) = 0$ a.s., and hence $x_\lambda' \partial g(u_{0,t-1}, \sigma_{t-1}^2; \theta_0)/\partial\lambda = 0$ a.s. By Assumption N5(iii), $x_\lambda = 0$, and hence we have proved that $\mathcal{I}(\theta_0)$ is positive definite.

Regarding the matrix $\mathcal{J}(\theta_0)$, note that $x'\mathcal{J}(\theta_0)\,x = 0$ now directly implies that

$$2E\left[\left(x_\mu' f_{\mu,t}(\theta_0)\right)^2\sigma_t^{-2}\right] + E\left[\left(x'h_{\theta,t}^*(\theta_0)\right)^2\sigma_t^{-4}\right] = 0.$$

This can only happen if $x_\mu' f_{\mu,t}(\theta_0) = 0$ a.s. and $x'h_{\theta,t}^*(\theta_0) = 0$ a.s. As above, this implies that $x = 0$. Hence also $\mathcal{J}(\theta_0)$ is positive definite. ■

Proof of Lemma D.4. First note that from the proof of Theorem 1 it can be seen that $\tilde{\theta}_T \to \theta_0$ a.s. (because $\liminf_{T\to\infty} \inf_{\theta\in B(\theta_0,\delta)^c} (L_T^*(\theta) - L_T^*(\theta_0))$ equals the sum of the last two terms on the minorant side of (22)). In the mean value expansion of $L_{\theta,T}^*(\theta)$ in (27) we therefore have $\dot{\theta}_{i,T} \to \theta_0$ a.s. as $T \to \infty$ $(i = 1, \ldots, m + l)$ which, together with the uniform convergence result for $L_{\theta\theta,T}^*(\theta)$ in Lemma D.2, yields $\dot{L}_{\theta\theta,T}^* \to \mathcal{J}(\theta_0)$ a.s. as $T \to \infty$. This and the invertibility of $\mathcal{J}(\theta_0)$ obtained from Lemma D.3 implies that, for all T sufficiently large, $\dot{L}_{\theta\theta,T}^*$ is also invertible (a.s.) and $\dot{L}_{\theta\theta,T}^{*-1} \to \mathcal{J}(\theta_0)^{-1}$ a.s. as $T \to \infty$ (see Lemma A.1 of Pötscher and Prucha (1991b)). Multiplying the mean value expansion (27) with the Moore-Penrose inverse $\dot{L}_{\theta\theta,T}^{*+}$ of $\dot{L}_{\theta\theta,T}^*$ (this inverse exists for all T) and rearranging we obtain

$$T^{1/2}(\tilde{\theta}_T - \theta_0) = (I - \dot{L}_{\theta\theta,T}^{*+}\dot{L}_{\theta\theta,T}^*)T^{1/2}(\tilde{\theta}_T - \theta_0) + \dot{L}_{\theta\theta,T}^{*+}T^{1/2}L_{\theta,T}^*(\tilde{\theta}_T) - \dot{L}_{\theta\theta,T}^{*+}T^{1/2}L_{\theta,T}^*(\theta_0) \text{ a.s.} \quad (44)$$

The first two terms on the right hand side of (44) converge to zero a.s. (more precisely, for all events ω on a set with probability one, there exists a $T(\omega)$ such that for all $T \geq T(\omega)$ the first two terms are identically equal to zero). For the first term, this follows from the fact that for all T sufficiently large $\dot{L}_{\theta\theta,T}^*$ is invertible (a.s.). For the second one, this holds because $\tilde{\theta}_T$ being a minimizer of $L_T^*(\theta)$ and θ_0 being an interior point of Θ_0 yield $L_{\theta,T}^*(\tilde{\theta}_T) = 0$ (a.s.) for all T sufficiently large. Furthermore, the eventual a.s. invertibility of $\dot{L}_{\theta\theta,T}^*$ also means that $\dot{L}_{\theta\theta,T}^{*+} - \mathcal{J}(\theta_0)^{-1} \to 0$ a.s. Hence, (44) becomes

$$T^{1/2}(\tilde{\theta}_T - \theta_0) = o_1(1) - (\mathcal{J}(\theta_0)^{-1} + o_2(1))T^{1/2}L_{\theta,T}^*(\theta_0) \text{ a.s.},$$

where $o_1(1)$ and $o_2(1)$ (a vector- and a matrix-valued process, respectively) converge to zero a.s. Combining this with the result of Lemma D.1 completes the proof. ■

Proof of Lemma D.5, remaining details. Using Assumption C4 and the inequality $|x^*y^* - xy| \leq |x^* - x|\,|y^*| + |x^* - x|\,|y^* - y| + |x^*|\,|y^* - y|$ for any conformable vectors one first obtains, a.s.,

$$\begin{aligned}
&\left|l_{\theta,t}^*(\theta) - l_{\theta,t}(\theta)\right| \\
= \;& \left|-\frac{h_{\theta,t}^*}{h_t^*}\left(\frac{u_t^2}{h_t^*} - 1\right) + \frac{h_{\theta,t}}{h_t}\left(\frac{u_t^2}{h_t} - 1\right) - 2\frac{f_{\theta,t}}{h_t^*}u_t + 2\frac{f_{\theta,t}}{h_t}u_t\right| \\
\leq \;& \left|\frac{h_{\theta,t}^*}{h_t^*} - \frac{h_{\theta,t}}{h_t}\right|\left|\frac{u_t^2}{h_t^*} - 1\right| + \left|\frac{h_{\theta,t}^*}{h_t^*} - \frac{h_{\theta,t}}{h_t}\right|\left|\frac{u_t^2}{h_t^*} - \frac{u_t^2}{h_t}\right| + \left|\frac{h_{\theta,t}^*}{h_t^*}\right|\left|\frac{u_t^2}{h_t^*} - \frac{u_t^2}{h_t}\right| + 2\left|f_{\theta,t}\right|\left|u_t\right|\left|\frac{1}{h_t^*} - \frac{1}{h_t}\right| \\
\leq \;& \left|\frac{h_{\theta,t}^*}{h_t^*} - \frac{h_{\theta,t}}{h_t}\right|\left[(\underline{g}^{-1}u_t^2 + 1) + u_t^2\underline{g}^{-2}\left|h_t^* - h_t\right|\right] + \left(\left|h_{\theta,t}^*\right|u_t^2\underline{g}^{-3} + 2\left|f_{\theta,t}\right|\left|u_t\right|\underline{g}^{-2}\right)\left|h_t^* - h_t\right|.
\end{aligned}$$

Then, making use of Lemma A.1, Hölder's inequality, and the norm inequality, one obtains the stated inequality. ■

Proof of (17). In this proof we assume $r = 4$, but retain the notation r for ease of comparison to previous results. It suffices to show that the four quantities in (16) are strongly consistent estimators of those in (15). Due to the strong consistency of $\hat{\theta}_T$, it suffices to prove that

$$\left|T^{-1}\sum_{t=1}^{T}A_t^{*(i)}-E\big[A_t^{*(i)}\big]\right|,\ \ i=1,\ldots,4,\quad\text{and}\quad\left|T^{-1}\sum_{t=1}^{T}\big(A_t^{*(i)}-A_t^{(i)}\big)\right|,\ \ i=1,\ldots,4,\quad(45)$$

converge to zero almost surely uniformly over Θ_0 as $T\to\infty$, where

$$A_t^{*(1)}=\frac{f_{\mu,t}}{h_t^{*1/2}}\frac{f'_{\mu,t}}{h_t^{*1/2}},\quad A_t^{*(2)}=\frac{h^*_{\theta,t}}{h^*_t}\frac{h^{*\prime}_{\theta,t}}{h^*_t},\quad A_t^{*(3)}=\frac{u_t^4}{h_t^{*2}}\frac{h^*_{\theta,t}}{h^*_t}\frac{h^{*\prime}_{\theta,t}}{h^*_t},\quad\text{and}\quad A_t^{*(4)}=\frac{u_t^3}{h_t^{*3/2}}\frac{f_{\mu,t}}{h_t^{*1/2}}\frac{h^{*\prime}_{\theta,t}}{h^*_t},$$

and $A_t^{(i)}$, $i=1,\ldots,4$, are defined similarly but with h_t^* and $h^*_{\theta,t}$ replaced with h_t and $h_{\theta,t}$.

Concerning the former four convergences in (45), these can be deduced from Theorem 2.7 of Straumann and Mikosch (2006) if $E\big[\sup_{\theta\in\Theta_0}|A_t^{*(i)}|\big]<\infty$, $i=1,\ldots,4$, holds. For $i=1$, this follows from the fact that $f_{\mu,t}$ is L_{2r}–dominated in Θ_0 (see the proof of Lemma C.1) and Assumption C4, whereas for $i=2$, this holds due to Assumption N4(ii). For $i=3$, the finiteness follows in view of Assumption C4, the fact that $\big\|\sup_{\theta\in\Theta}|u_t|\big\|_{2r}<\infty$ (see the proof of Proposition 1), and Assumption N4(ii). For $i=4$, this follows from the aforementioned facts that $f_{\mu,t}$ is L_{2r}–dominated in Θ_0 and $\big\|\sup_{\theta\in\Theta}|u_t|\big\|_{2r}<\infty$ as well as Assumptions C4 and N4(ii).

Now consider the latter four convergences in (45). For $i=1$, use Assumption C4 to obtain

$$|A_t^{*(1)}-A_t^{(1)}|=|h_t^{*-1}f_{\mu,t}f'_{\mu,t}-h_t^{-1}f_{\mu,t}f'_{\mu,t}|\le\underline{g}^{-2}|f_{\mu,t}f'_{\mu,t}||h^*_t-h_t|.$$

Thus, by the Cauchy-Schwartz inequality, the aforementioned L_{2r}–dominance of $f_{\mu,t}$, and (21),

$$\left\|\sup_{\theta\in\Theta_0}|A_t^{*(1)}-A_t^{(1)}|\right\|_{r/2}\le\underline{g}^{-2}\left\|\sup_{\theta\in\Theta_0}|f_{\mu,t}f'_{\mu,t}|\right\|_r\left\|\sup_{\theta\in\Theta_0}|h^*_t-h_t|\right\|_r\le C\kappa^t$$

for some finite C. The required convergence for $i=1$ now follows from Lemma A.2.

The cases $i=2$, 3, and 4 can be handled in a similar way. We only note that for $i=2$,

$$|A_t^{*(2)}-A_t^{(2)}|\le2|h^*_{\theta,t}/h^*_t||h^*_{\theta,t}/h^*_t-h_{\theta,t}/h_t|+|h^*_{\theta,t}/h^*_t-h_{\theta,t}/h_t|^2,$$

and thus by Lemma A.1, Cauchy-Schwartz inequality, Assumption N4(ii), and inequality (30),

$$\begin{aligned}&\left\|\sup_{\theta\in\Theta_0}|A_t^{*(2)}-A_t^{(2)}|\right\|_{r/8}\\ \le\ &\Delta_{r/8,2}\left(2\left\|\sup_{\theta\in\Theta_0}|h^*_{\theta,t}/h^*_t|\right\|_{r/4}\left\|\sup_{\theta\in\Theta_0}|h^*_{\theta,t}/h^*_t-h_{\theta,t}/h_t|\right\|_{r/4}+\left\|\sup_{\theta\in\Theta_0}|h^*_{\theta,t}/h^*_t-h_{\theta,t}/h_t|^2\right\|_{r/8}\right)\\ \le\ &C\big(\max\{t,t^{4/r}\}\kappa^t+\max\{t^2,t^{8/r}\}\kappa^{2t}\big),\end{aligned}$$

for some finite C. For $i=3$, using the inequality $|x^*y^*-xy|\le|x^*-x|\,|y^*|+|x^*|\,|y^*-y|+|x^*-x|\,|y^*-y|$ for any conformable vectors,

$$|A_t^{*(3)}-A_t^{(3)}|\le\left|\frac{u_t^4}{h_t^{*2}}-\frac{u_t^4}{h_t^2}\right|\left|\frac{h^*_{\theta,t}}{h^*_t}\frac{h^{*\prime}_{\theta,t}}{h^*_t}\right|+\left|\frac{u_t^4}{h_t^{*2}}-\frac{u_t^4}{h_t^2}\right|\left|\frac{h^*_{\theta,t}}{h^*_t}\frac{h^{*\prime}_{\theta,t}}{h^*_t}-\frac{h_{\theta,t}}{h_t}\frac{h'_{\theta,t}}{h_t}\right|+\left|\frac{u_t^4}{h_t^{*2}}\right|\left|\frac{h^*_{\theta,t}}{h^*_t}\frac{h^{*\prime}_{\theta,t}}{h^*_t}-\frac{h_{\theta,t}}{h_t}\frac{h'_{\theta,t}}{h_t}\right|,$$

where the term $|A_t^{*(2)} - A_t^{(2)}|$ reappears in the two last terms and

$$|u_t^4/h_t^{*2} - u_t^4/h_t^2| = u_t^4 |h_t/h_t^{*2}h_t - h_t^*/h_t^{*2}h_t + h_t/h_t^*h_t^2 - h_t^*/h_t^*h_t^2| \leq 2\underline{g}^{-3}u_t^4|h_t^* - h_t|.$$

The required subsequent steps are similar to those used above and are omitted. For $i = 4$, the term $|A_t^{*(4)} - A_t^{(4)}|$ can be bounded from above by

$$\left|\frac{u_t^3}{h_t^{*3/2}} - \frac{u_t^3}{h_t^{3/2}}\right| \left|\frac{f_{\mu,t}}{h_t^{*1/2}}\frac{h_{\theta,t}^{*\prime}}{h_t^*}\right| + \left|\frac{u_t^3}{h_t^{*3/2}} - \frac{u_t^3}{h_t^{3/2}}\right| \left|\frac{f_{\mu,t}}{h_t^{*1/2}}\frac{h_{\theta,t}^{*\prime}}{h_t^*} - \frac{f_{\mu,t}}{h_t^{1/2}}\frac{h_{\theta,t}^{\prime}}{h_t}\right| + \left|\frac{u_t^3}{h_t^{*3/2}}\right| \left|\frac{f_{\mu,t}}{h_t^{*1/2}}\frac{h_{\theta,t}^{*\prime}}{h_t^*} - \frac{f_{\mu,t}}{h_t^{1/2}}\frac{h_{\theta,t}^{\prime}}{h_t}\right|,$$

where, using the mean value theorem for the function $x^{-3/2}$,

$$\left|u_t^3/h_t^{*3/2} - u_t^3/h_t^{3/2}\right| \leq |u_t^3| \, \frac{3}{2}\underline{g}^{-5/2}|h_t^* - h_t|.$$

We again omit the remaining steps, and only point out that it suffices to consider the norm $\left\|\sup_{\theta\in\Theta_0}|A_t^{*(4)} - A_t^{(4)}|\right\|_p$ with any $p > 0$. Thus, we have justified (17). ■

Verification of Assumption DGP in Example 3, remaining details. What remains to show is that the conditions in (a) imply that Assumptions 1–4, 5(b), and 6 of Meitz and Saikkonen (2008b) hold so that from Theorem 1 of that paper we can conclude that Assumption DGP holds. To see this, note first that conditions (a.i) and (a.ii) imply that Assumption 1 of Meitz and Saikkonen (2008b) holds, whereas the conditions imposed on the function F in (a.vi) and the assumed range of F imply Assumption 2 of the same paper. That Assumption 3 of Meitz and Saikkonen (2008b) holds follows from the discussion given in Section 4 of that paper and condition (a.iii). Finally, (a.iv), (a.v), and the conditions assumed about the function G in (a.vi) and its range imply that the model satisfies the assumptions required for the model for conditional variance in Proposition 1 of Meitz and Saikkonen (2008b). Of the two alternative cases in that proposition, (a) and (b), the latter is relevant, and it follows that Theorem 1 of Meitz and Saikkonen (2008b) applies with some $r_0 \in (0, 1)$. Thus, Assumption DGP holds with $r = r_0$. ■

Verification of Assumptions for asymptotic normality in Example 3, remaining details. What remains is to verify Assumption N4(ii). In what follows, we assume that $\theta \in \Theta_0$. Moreover, without loss of generality we may assume Θ_0 is small enough to ensure that $\theta \in \Theta_0$ implies $0 < \underline{\omega} \leq \omega \leq \overline{\omega} < \infty$, $0 < \underline{\alpha}_1 \leq \alpha_1 \leq \overline{\alpha}_1 < \infty$, $0 < \underline{\alpha}_2 \leq \alpha_2 \leq \overline{\alpha}_2 < \infty$, $0 < \underline{\beta} \leq \beta \leq \overline{\beta} < 1$, $\varphi \in N(\varphi_0)$, and $\gamma \in N(\gamma_0)$. Now, for the first norm in Assumption N4(ii) concerning the vector $h_{\theta,t}^*/h_t^*$, recall that in the present case $h_t^* = \omega + (\alpha_1 + \alpha_2 G(u_{t-1}; \gamma))\, u_{t-1}^2 + \beta h_{t-1}^*$ (where the argument θ has been suppressed from h_t^* and u_t) and, in the notation of Section 4, $h_{\theta,t}^* = g_{\theta,t}^* - g_{u,t}^* f_{\theta,t-1} + g_{h,t}^* h_{\theta,t-1}^*$ (see equations (7) and (11)). Partitioning $h_{\theta,t}^*$ as $h_{\theta,t}^* = (h_{\mu,t}^*, h_{\lambda,t}^*)$ we obtain $h_{\mu,t}^* = -g_{u,t}^* f_{\mu,t-1} + \beta h_{\mu,t-1}^*$ and $h_{\lambda,t}^* = g_{\lambda,t}^* + \beta h_{\lambda,t-1}^*$ as immediate consequences of the definitions. Because $\beta \leq \bar{\beta} < 1$ by assumption, $h^*_{\mu,t}$ and $h^*_{\lambda,t}$ have the representations

$$h^*_{\mu,t} = -\sum_{j=0}^{\infty} \beta^j g^*_{u,t-j} f_{\mu,t-j-1} \quad \text{and} \quad h^*_{\lambda,t} = \sum_{j=0}^{\infty} \beta^j g^*_{\lambda,t-j}, \tag{46}$$

where the infinite sums converge due to Lemmas A.2 and C.1. By straightforward derivation, $g^*_{u,t} = 2\left(\alpha_1 + \alpha_2 G(u_{t-1};\gamma)\right) u_{t-1} + \alpha_2 G_u\left(u_{t-1};\gamma\right) u^2_{t-1}$, whereas the vector $g^*_{\lambda,t}$ has components

$$1,\ u^2_{t-1},\ G(u_{t-1};\gamma)u^2_{t-1},\ h^*_{t-1},\ \text{and } \alpha_2 G_\gamma(u_{t-1};\gamma)u^2_{t-1}. \tag{47}$$

Because the range of G is $[0,1]$, and $G_u\left(u;\gamma\right) u^2$ and $G_\gamma(u;\gamma)$ are bounded in absolute value uniformly over $\mathbb{R} \times N(\gamma_0)$ by conditions (c.iii) and (c.vi), the finiteness of $\left\|\sup_{\theta\in\Theta_0}|h^*_{\mu,t}|/h^*_t\right\|_4$ and $\left\|\sup_{\theta\in\Theta_0}|h^*_{\lambda,t}|/h^*_t\right\|_4$, and hence of the first norm in Assumption N4, follows if we show that

$$\left\|\sup_{\theta\in\Theta_0} \sum_{j=0}^{\infty} \beta^j a^{(i)}_{t-1-j}/h^*_t\right\|_4 < \infty, \quad i = 1,\ldots,4, \tag{48}$$

where $a^{(1)}_t = u^2_t$, $a^{(2)}_t = h^*_t$, $a^{(3)}_t = |u_t||f_{\mu,t}|$, and $a^{(4)}_t = |f_{\mu,t}|$. To show this, first express h^*_t as

$$h^*_t = \sum_{k=0}^{\infty} \beta^k \left(\omega + \left(\alpha_1 + \alpha_2 G(u_{t-1-k};\gamma)\right) u^2_{t-1-k}\right), \tag{49}$$

where the infinite sum converges due to Lemma A.2 and the result $\left\|\sup_{\theta\in\Theta}|u_t|\right\|_{2r} < \infty$ obtained in the proof of Proposition 1. Because $\omega \geq \underline{\omega} > 0$, $\alpha_1 \geq \underline{\alpha}_1 > 0$, $\alpha_2 \geq \underline{\alpha}_2 > 0$, and $\beta \geq \underline{\beta} > 0$,

$$h^*_t \geq \sum_{k=0}^{\infty} \beta^k \left(\omega + \alpha_1 u^2_{t-1-k}\right) \geq \underline{\omega} + \beta^j \underline{\alpha}_1 u^2_{t-1-j} \tag{50}$$

for any $j \geq 0$. Now, considering (48) with $i = 1$ and making use of (50) and the fact that $x/(1+x) \leq x^s$ for all $x \geq 0$ and any $s \in (0,1)$ (cf. Francq and Zakoïan (2004), above their equation (4.25)), we obtain that, for any $j \geq 0$ and $s \in (0,1)$,

$$\frac{\beta^j u^2_{t-1-j}}{h^*_t} \leq \underline{\alpha}_1^{-1} \frac{\beta^j \underline{\alpha}_1 u^2_{t-1-j}/\underline{\omega}}{1 + \beta^j \underline{\alpha}_1 u^2_{t-1-j}/\underline{\omega}} \leq \underline{\alpha}_1^{s-1} \underline{\omega}^{-s} \beta^{sj} \left|u_{t-1-j}\right|^{2s} \leq \underline{\alpha}_1^{s-1} \underline{\omega}^{-s} \bar{\beta}^{sj} \sup_{\theta\in\Theta_0} \left|u_{t-1-j}\right|^{2s}. \tag{51}$$

As was noted above, $\left\|\sup_{\theta\in\Theta}|u_t|\right\|_{2r} < \infty$, or $\left\|\sup_{\theta\in\Theta}|u_t|\right\|_4 < \infty$ when $r = 2$ is assumed. Thus, choosing $s \leq 1/2$ and making use of the norm inequality we obtain $\left\|\sup_{\theta\in\Theta_0}|u_t|^{2s}\right\|_4 \leq \left\|\sup_{\theta\in\Theta_0}|u_t|\right\|_4^{2s}$. Using this fact, (51), and Minkowski's inequality we find that

$$\left\|\sup_{\theta\in\Theta_0} \sum_{j=0}^{\infty} \beta^j u^2_{t-1-j}/h^*_t\right\|_4 \leq \underline{\alpha}_1^{s-1} \underline{\omega}^{-s} \sum_{j=0}^{\infty} \bar{\beta}^{sj} \left\|\sup_{\theta\in\Theta_0} \left|u_{t-1-j}\right|\right\|_4^{2s},$$

where the majorant side is finite. Hence we have established (48) with $i = 1$.

Now consider (48) with $i = 2$ and conclude from (49) and (50) that

$$\frac{h^*_{t-1-j}}{h^*_t} \leq \sum_{k=0}^{\infty} \beta^k \frac{\omega + (\alpha_1 + \alpha_2 G(u_{t-2-j-k};\gamma))\, u^2_{t-2-j-k}}{\underline{\omega} + \beta^{j+k+1}\underline{\alpha}_1 u^2_{t-2-j-k}}$$

for any $j \geq 0$. Because $\omega \leq \overline{\omega}$ and $\alpha_1 + \alpha_2 G(u_{t-2-j-k};\gamma) \leq C$ for some finite C, we have

$$\frac{h^*_{t-1-j}}{h^*_t} \leq \frac{\overline{\omega}}{\underline{\omega}} \sum_{k=0}^{\infty} \bar{\beta}^k + C \sum_{k=0}^{\infty} \beta^k \frac{u^2_{t-2-j-k}}{\underline{\omega} + \beta^{j+k+1}\underline{\alpha}_1 u^2_{t-2-j-k}}.$$

Hence, by arguments similar to those used to derive (51) we have, for any $j \geq 0$ and $s \in (0,1)$,

$$\begin{aligned}
\beta^j \frac{h^*_{t-1-j}}{h^*_t} &\leq \beta^j \frac{\overline{\omega}}{\underline{\omega}} \sum_{k=0}^{\infty} \bar{\beta}^k + \frac{C}{\underline{\alpha}_1 \beta} \sum_{k=0}^{\infty} \frac{\beta^{j+k+1}\underline{\alpha}_1 u^2_{t-2-j-k}/\underline{\omega}}{1 + \beta^{j+k+1}\underline{\alpha}_1 u^2_{t-2-j-k}/\underline{\omega}} \\
&\leq \beta^j \frac{\overline{\omega}}{\underline{\omega}} \left(1 - \bar{\beta}\right)^{-1} + \frac{C\underline{\alpha}_1^{s-1}}{\underline{\omega}^s \underline{\beta}} \sum_{k=0}^{\infty} \beta^{(j+k+1)s} \left|u_{t-2-j-k}\right|^{2s} \\
&\leq \bar{\beta}^j \frac{\overline{\omega}}{\underline{\omega}} \left(1 - \bar{\beta}\right)^{-1} + \frac{C\underline{\alpha}_1^{s-1}\bar{\beta}^s}{\underline{\omega}\underline{\beta}} \bar{\beta}^{js} \sum_{k=0}^{\infty} \bar{\beta}^{ks} \sup_{\theta\in\Theta_0} \left|u_{t-2-j-k}\right|^{2s}.
\end{aligned}$$

Choosing $s \leq 1/2$ and using Minkowski's inequality and the norm inequality in the same way as in the case $i = 1$ we find that the norm in (48) is finite when $i = 2$.

Next consider (48) with $i = 3$. In view of (50) and the inequality $x/(1+x^2) \leq 1$ (cf. Francq and Zakoïan (2004), above their equation (4.49)) we have, for any $j \geq 0$,

$$\begin{aligned}
\beta^j \left|u_{t-1-j}\right| \left|f_{\mu,t-1-j}\right| / h^*_t &\leq \frac{(\beta^j u^2_{t-1-j})^{1/2}}{\underline{\omega} + \beta^j \underline{\alpha}_1 u^2_{t-1-j}} \beta^{j/2} \left|f_{\mu,t-j-1}\right| \\
&\leq (\underline{\alpha}_1\underline{\omega})^{-1/2} \frac{(\beta^j \underline{\alpha}_1 u^2_{t-1-j}/\underline{\omega})^{1/2}}{1 + \beta^j \underline{\alpha}_1 u^2_{t-1-j}/\underline{\omega}} \bar{\beta}^{j/2} \left|f_{\mu,t-j-1}\right| \\
&\leq (\underline{\alpha}_1\underline{\omega})^{-1/2} \bar{\beta}^{j/2} \left|f_{\mu,t-j-1}\right|.
\end{aligned}$$

As in the case $i = 1$, Minkowski's inequality shows that (48) holds with $i = 3$ if $\left\|\sup_{\theta\in\Theta_0}|f_{\mu,t}|\right\|_4 < \infty$. To verify this, calculate the partial derivatives of $f\left(y_{t-1},\ldots,y_{t-p};\mu\right)$ as

$$1,\ y_{t-1},\ldots,y_{t-p},\ \ (1, y_{t-1},\ldots,y_{t-p})\, F(y_{t-d};\varphi),\ \text{and}\ (\psi_0 + \sum_{j=1}^{p} \psi_j y_{t-j}) F_{\varphi}(y_{t-d};\varphi). \qquad (52)$$

Because the range of F is $[0,1]$ and the partial derivatives of F are bounded uniformly over $\mathbb{R}\times N(\varphi_0)$ by condition (c.iii), we have $|f_{\mu,t}| \leq C(1 + \sum_{j=1}^{p}|y_{t-j}|)$ for some finite C. Thus, the desired result follows because $E[y_t^4] < \infty$ in view of Assumption DGP and the fact that $r = 2$.

For (48) with $i = 4$ it suffices to note that $\beta^j|f_{\mu,t-1-j}|/h^*_t \leq \underline{g}^{-1}\bar{\beta}^j|f_{\mu,t-1-j}|$ by Assumption C4, and hence the result follows as in the case $i = 3$.

Now consider the latter norm in Assumption N4(ii). Recall from Section 4 that

$$h^*_{\theta\theta,t} = \alpha^*_{\theta\theta,t} + \beta^*_t h^*_{\theta\theta,t-1} + \gamma^*_{\theta,t} h^{*\prime}_{\theta,t-1} + h^*_{\theta,t-1}\gamma^{*\prime}_{\theta,t} + \delta^*_t h^*_{\theta,t-1} h^{*\prime}_{\theta,t-1},$$

where $\alpha^*_{\theta\theta,t}$, β^*_t, $\gamma^*_{\theta,t}$, and δ^*_t are as in (7)–(8) but with h_t throughout replaced with h^*_t. As already noticed, $\beta^*_t = g^*_{h,t} = \beta$, which implies that $g^*_{hh,t} = 0$ and $g^*_{uh,t} = 0$. Moreover, only one element of $g^*_{\theta h,t}$ is nonzero, namely the one related to the component β of θ for which the resulting partial derivative is unity. Thus, $\delta^*_t = 0$, $\gamma^*_{\theta,t} = \gamma^*_\theta$ is independent of t, and we get $h^*_{\theta\theta,t} = \alpha^*_{\theta\theta,t} + \gamma^*_\theta h^{*\prime}_{\theta,t-1} + h^*_{\theta,t-1}\gamma^{*\prime}_\theta + \beta h^*_{\theta\theta,t-1}$, giving the representation

$$h^*_{\theta\theta,t} = \sum_{j=0}^{\infty} \beta^j \alpha^*_{\theta\theta,t-j} + \sum_{j=0}^{\infty} \beta^j \gamma^*_\theta h^{*\prime}_{\theta,t-1-j} + \sum_{j=0}^{\infty} \beta^j h^*_{\theta,t-1-j}\gamma^{*\prime}_\theta$$

(the infinite sums converge due to Lemmas A.2 and C.1 and Proposition 2). This, and the definition of $\alpha^*_{\theta\theta,t}$, show that for $\left\|\sup_{\theta\in\Theta_0}|h^*_{\theta\theta,t}|/h^*_t\right\|_2 < \infty$ it suffices to establish that

$$\left\| \sup_{\theta\in\Theta_0} \sum_{j=0}^{\infty} \beta^j a^{(i)}_{t-1-j}/h^*_t \right\|_2 < \infty, \quad i = 5, \ldots, 9, \tag{53}$$

where $a^{(5)}_t = |g^*_{\theta\theta,t+1}|$, $a^{(6)}_t = |g^*_{uu,t+1}||f_{\theta,t}|^2$, $a^{(7)}_t = |g^*_{u\theta,t+1}||f_{\theta,t}|$, $a^{(8)}_t = |g^*_{u,t+1}||f_{\theta\theta,t}|$, and $a^{(9)}_t = |h^*_{\theta,t}|$.

Because the details of verifying (53) are similar to those already used to deduce (48), we only sketch the required steps. For (53) with $i = 5$, note that because $g^*_{\mu,t+1} = 0$, also $g^*_{\mu\mu,t+1} = 0$ and $g^*_{\lambda\mu,t+1} = 0 = g^{*\prime}_{\mu\lambda,t+1}$. Moreover, by direct calculation, it can be seen that the only nonnull elements of $g^*_{\lambda\lambda,t+1}$ are $G_\gamma(u_t;\gamma)u_t^2$ and $\alpha_2 G_{\gamma\gamma}(u_t;\gamma)u_t^2$. Therefore, $|g^*_{\theta\theta,t+1}|$ is dominated by Cu_t^2 $(C < \infty)$. Arguments similar to those used to show (48) with $i = 1$ can now be applied to verify (53) with $i = 5$ (we omit the details). Next, for (53) with $i = 6$, straightforward differentiation gives $g^*_{uu,t+1} = 2(\alpha_1 + \alpha_2 G(u_t;\gamma)) + 4\alpha_2 G_u(u_t;\gamma)u_t + \alpha_2 G_{uu}(u_t;\gamma)u_t^2$. By condition (c.vi), $\sup_{\theta\in\Theta_0}|g^*_{uu,t+1}|$ is bounded, and therefore arguments used for (48) with $i = 4$ can be used to obtain the desired result. For (53) with $i = 7$, consider $g^*_{u\theta,t+1}$ and note that $g^*_{u\mu,t+1} = 0$ whereas the nonnull elements of the matrix $g^*_{u\lambda,t+1}$ are $2u_t$, $2G(u_t;\gamma)u_t + G_u(u_t;\gamma)u_t^2$, and $2\alpha_2 G_\gamma(u_t;\gamma)u_t + \alpha_2 G_{u\gamma}(u_t;\gamma)u_t^2$. By conditions (c.iii) and (c.vi), $|g^*_{u\theta,t+1}|$ is dominated by $C(1 + |u_t|)$ $(C < \infty)$, and arguments already used to verify (48) with $i = 3$ can be applied to deduce (53) with $i = 7$. Now consider (53) with $i = 8$. By (38) and direct calculation, the nonnull elements of $f_{\theta\theta,t-1}$ are $(1, y_{t-1}, \ldots, y_{t-p})\, F_\varphi(y_{t-d};\varphi)$ and $(\psi_0 + \sum_{j=1}^p \psi_j y_{t-j})F_{\varphi\varphi}(y_{t-d};\varphi)$. Thus, similarly to $\sup_{\theta\in\Theta_0}|f_{\theta,t-1}| = \sup_{\theta\in\Theta_0}|f_{\mu,t-1}|$ also $\sup_{\theta\in\Theta_0}|f_{\theta\theta,t-1}|$ is dominated by a term of the form $C(1+\sum_{j=1}^p|y_{t-j}|)$ with a finite C. Arguments used for (48) with $i = 3$ and 4 can now be used to deduce (53). Finally, for (53) with $i = 9$, recall that we have shown that $\left\|\sup_{\theta\in\Theta_0}|h^*_{\theta,t}|/h^*_t\right\|_4$ is finite, and thus Minkowski's inequality gives $\left\|\sup_{\theta\in\Theta_0}\sum_{j=0}^\infty \beta^j|h^*_{\theta,t-1-j}|/h^*_t\right\|_2 < \infty$. This completes the verification of (53) and that of $\left\|\sup_{\theta\in\Theta_0}|h^*_{\theta\theta,t}|/h^*_t\right\|_2 < \infty$. Thus, N4(ii) holds. ■

Additional References

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