Supplementary Appendix to "Maximum likelihood estimation of a noninvertible ARMA model with autoregressive conditional heteroskedasticity" by Mika Meitz and Pentti Saikkonen (contains additional proofs, not to be published)

Proof of Lemma A.1. (i) As Θ_a is open, we can choose a small neighborhood $N(\theta_a^\bullet)$ whose closure $\overline{N(\theta_a^\bullet)}$ is contained in Θ_a. By continuity of $a(\cdot;\cdot)$, compactness of $\{z : |z| \leq 1\} \times \overline{N(\theta_a^\bullet)}$, and the condition $a(z;\theta_a) \neq 0$ for $|z| \leq 1$ and $\theta_a \in \Theta_a$, we can find a positive $\delta(\theta_a^\bullet)$ such that $a(z;\theta_a) \neq 0$ for $|z| \leq 1 + 2\delta(\theta_a^\bullet)$ and $\theta_a \in \overline{N(\theta_a^\bullet)}$. That the inverse $a(z;\theta_a)^{-1}$ has the stated Laurent series expansion for each $\theta_a \in \overline{N(\theta_a^\bullet)}$ now follows from well-known results.

To prove the latter statement, for each $\theta_a \in \overline{N(\theta_a^\bullet)}$, factor the polynomial $a(z;\theta_a)$ into first order terms as $a(z;\theta_a) = \prod_{p=1}^{P}(1 - z/z_p(\theta_a))$, where $z_p(\theta_a)$, $p = 1,\ldots,P$, are the roots of $a(z;\theta_a)$. The preceding discussion shows that $|z_p(\theta_a)| > 1 + 2\delta(\theta_a^\bullet)$ for $p = 1,\ldots,P$ and all $\theta_a \in \overline{N(\theta_a^\bullet)}$. Thus, $\sup_{\theta_a \in \overline{N(\theta_a^\bullet)},\ p=1,\ldots,P} |z_p(\theta_a)|^{-1} \leq (1 + \delta(\theta_a^\bullet))^{-1} \stackrel{def}{=} \rho < 1$. Next note that $a(z;\theta_a)^{-1}$ can be expressed as

$$a(z;\theta_a)^{-1} = \prod_{p=1}^{P}(1 - z/z_p(\theta_a))^{-1} = \prod_{p=1}^{P}\sum_{j=0}^{\infty} z_p(\theta_a)^{-j} z^j$$

because each of the involved first order terms has a Laurent series expansion valid on the annulus mentioned above. Therefore, the coefficient $\psi_j^{(a)}(\theta_a)$ can be expressed as

$$\psi_j^{(a)}(\theta_a) = \sum_{\substack{j_1,\ldots,j_P \geq 0 \\ j_1+\ldots+j_P = j}} z_1(\theta_a)^{-j_1} \cdots z_P(\theta_a)^{-j_P}.$$

Here each of the summands is bounded from above by ρ^j, and this holds for all $\theta_a \in \overline{N(\theta_a^\bullet)}$. Moreover, the number of terms in this summation is $\binom{j+P-1}{j}$. Therefore,

$$\sup_{\theta_a \in \overline{N(\theta_a^\bullet)}} \psi_j^{(a)}(\theta_a) \leq \binom{j+P-1}{j}\rho^j = \binom{j+P-1}{j}\left(\frac{\rho}{\rho_a}\right)^j \rho_a^j,$$

where ρ_a is chosen between ρ and 1 and, for fixed P, the binomial is a $(P-1)$th order polynomial of j. Because $\binom{j+P-1}{j}(\rho/\rho_a)^j \to 0$ as $j \to \infty$, we may choose a finite C (independent of θ_a) such that, for all j,

$$\sup_{\theta_a \in \overline{N(\theta_a^\bullet)}} \psi_j^{(a)}(\theta_a) \leq C\rho_a^j,$$

completing the proof of (i). Proof of part (ii) is analogous, so details are omitted. ■

Proof of Lemma A.2. To show that the infinite sum is well defined, fix $\phi \in \Phi$ and first consider the one-sided sum with $j \geq 1$. Choosing a γ such that $1 < \gamma < \rho^{-1}$, we have $\gamma\rho < 1$, and hence the sequence $\gamma^j |\kappa_j(\phi) X_{t-j}(\phi)|$ converges to zero in L_r–norm as $j \to \infty$. To see this, note that, due to the assumed boundedness of $\kappa_j(\phi)$, $\gamma^j |\kappa_j(\phi) X_{t-j}(\phi)| \leq C(\gamma\rho)^j |X_{t-j}(\phi)|$ and, therefore,

$$\left\| \gamma^j |\kappa_j(\phi) X_{t-j}(\phi)| \right\|_r \leq C(\gamma\rho)^j \|X_{t-j}(\phi)\|_r,$$

where the majorant converges to zero as $j \to \infty$. Now by Lemma A.2 in Meitz and Saikkonen (2011), we have

$$\sum_{j=1}^{\infty} |\kappa_j(\phi) X_{t-j}(\phi)| < \infty \text{ a.s.}$$

from which it follows that $\sum_{j=1}^{\infty} \kappa_j(\phi) X_{t-j}(\phi)$ is finite with probability one as well. Exactly the same argument shows that also the sum $\sum_{j=-\infty}^{-1} \kappa_j(\phi) X_{t-j}(\phi)$ is finite (and so is $\kappa_0(\phi) X_t(\phi)$), and therefore $\sum_{j=-\infty}^{\infty} \kappa_j(\phi) X_{t-j}(\phi)$ converges with probability one.

To prove the last statement, define $x_j = \sup_{\phi\in\Phi} |\kappa_j(\phi) X_{t-j}(\phi)|$ for $j \geq 1$, and following exactly the same argument as above, conclude that

$$\|x_j\|_r \leq C\rho^j \left\| \sup_{\phi\in\Phi} |X_{t-j}(\phi)| \right\|_r \leq C'\rho^j$$

for some finite C'. As in the proof of Lemma A.2 in Meitz and Saikkonen (2011), the result $\left\|\sum_{j=1}^{\infty} \sup_{\phi\in\Phi} |\kappa_j(\phi) X_{t-j}(\phi)|\right\|_r < \infty$ now follows. Repeating the argument with the other one-sided sum, $\left\|\sum_{j=-\infty}^{\infty} \sup_{\phi\in\Phi} |\kappa_j(\phi) X_{t-j}(\phi)|\right\|_r < \infty$. To conclude the proof, it remains to note that this norm is an upper bound to $\left\|\sup_{\phi\in\Phi} |Y_t(\phi)|\right\|_r$. ■

Proof of Lemma A.3. Denoting $\tilde{Z}_i = (\ldots, Z_{i-1}, Z_{i+1}, \ldots)$ allows us to write $h(\ldots, Z_{i-1}, Z_i, Z_{i+1}, \ldots) = h_*(Z_i, \tilde{Z}_i)$, where Z_i and $\tilde{Z}_i$ are independent. Denoting the distribution of $(Z_i, \tilde{Z}_i)$ with $P_{(Z_i,\tilde{Z}_i)} = P_{Z_i} \times P_{\tilde{Z}_i}$ and applying Fubini's theorem,

$$E[Y] = E[h_*(Z_i, \tilde{Z}_i)] = \iint h_*(z_i, \tilde{z}_i) P_{(Z_i,\tilde{Z}_i)}(dz_i, d\tilde{z}_i) = \int \left(\int h_*(z_i, \tilde{z}_i) P_{Z_i}(dz_i) \right) P_{\tilde{Z}_i}(d\tilde{z}_i) = 0,$$

because the inner integral equals zero due to the assumed oddness of h. ■

Proof of Lemma A.4. Follows from the Functional Central Limit Theorem of Scott (1973, Theorem 3), because condition (41) therein is satisfied due to the assumption that $\{Z_t, \mathcal{F}_t^{\epsilon}\}$ is an L_2–mixingale with size -1. ■

Proof of Lemma B.1. If $a = 0$, the result is immediate, so assume $a \geq 1$. In this case, the term η_{t+a}^2 is independent of the remaining terms with $E\left[\eta_{t+a}^2\right] = 1$, and can thus be dropped from what follows. Noting that σ_{t+a}^2 can be expressed as

$$\sigma_{t+a}^2 = \mathbf{1}' \left(\mathbf{I}_R + \sum_{k=1}^{a-1} \prod_{l=1}^{k} \Pi_{t+a-l} \right) \varpi + \mathbf{1}' \left(\prod_{l=1}^{a} \Pi_{t+a-l} \right) X_t$$

the term $H_{t-1} g(\eta_t) \sigma_{t+a}^2 / \sigma_t^2$ can be written as

$$H_{t-1} g(\eta_t) \mathbf{1}' \left(\mathbf{I}_R + \sum_{k=1}^{a-1} \prod_{l=1}^{k} \Pi_{t+a-l} \right) \varpi / \sigma_t^2 + H_{t-1} g(\eta_t) \mathbf{1}' \left(\prod_{l=1}^{a} \Pi_{t+a-l} \right) X_t / \sigma_t^2. \tag{31}$$

In the former term in (31), the terms Π_{t+a-1}, $\ldots$, Π_{t+1} (if present) are independent of the remaining terms and of each other, with their product having expected value Π^{a-1}. Therefore this term has the same expected value as

$$\omega_0 \frac{H_{t-1}}{\sigma_t^2} g(\eta_t) \sum_{k=0}^{a-1} c_k$$

(and if $E\left[g(\eta_t)\right] = 0$, this term has expectation zero). Now consider the latter term in (31). The terms Π_{t+a-1}, $\ldots$, Π_{t+1} (if present) are independent of the remaining terms and of each other, with their product having expected value Π^{a-1}. The term $g(\eta_t)\Pi_t$ is also independent of the remaining terms with $E\left[g(\eta_t)\Pi_t\right]$ being a matrix with (1,1) element equal to $\alpha_{0,1}E\left[g(\eta_t)\eta_t^2\right]$, (2,1) element equal to $E\left[g(\eta_t)\eta_t^2\right]$, and the remaining elements being equal to $E\left[g(\eta_t)\right]$ times the corresponding element in Π. Thus, $E\left[g(\eta_t)\Pi_t\right]$ can be expressed as $\Pi\left\{E\left[g(\eta_t)\right]\mathbf{I}_R + \left(E\left[g(\eta_t)\eta_t^2\right] - E\left[g(\eta_t)\right]\right)\mathbf{1}\mathbf{1}'\right\}$. (In the case $R = 1$, $E\left[g(\eta_t)\Pi_t\right]$ is a scalar but nevertheless has the derived expression.) Therefore, the latter term in (31) has the same expected value as

$$\begin{aligned}
& H_{t-1}\mathbf{1}'\Pi^a\left\{E\left[g(\eta_t)\right]\mathbf{I}_R + \left(E\left[g(\eta_t)\eta_t^2\right] - E\left[g(\eta_t)\right]\right)\mathbf{1}\mathbf{1}'\right\}X_t/\sigma_t^2 \\
= \;& H_{t-1}\mathbf{1}'\Pi^a E\left[g(\eta_t)\right]X_t/\sigma_t^2 + H_{t-1}\left(E\left[g(\eta_t)\eta_t^2\right] - E\left[g(\eta_t)\right]\right)\mathbf{1}'\Pi^a\mathbf{1}\mathbf{1}'X_t/\sigma_t^2 \\
= \;& H_{t-1}\mathbf{1}'\Pi^a E\left[g(\eta_t)\right]X_t/\sigma_t^2 + H_{t-1}\left(E\left[g(\eta_t)\eta_t^2\right] - E\left[g(\eta_t)\right]\right)c_a \\
= \;& c_a H_{t-1}E\left[g(\eta_t)\eta_t^2\right] + E\left[g(\eta_t)\right]\left(\mathbf{1}'\Pi^a H_{t-1}X_t/\sigma_t^2 - c_a H_{t-1}\right).
\end{aligned}$$

This completes the proof. ■

Proof of Lemma C.1. Part (i) follows from the first dominance condition in Assumption 4(iv) which also implies part (ii) because, using continuity, we can find a finite constant C such that

$$\frac{f_{\eta,x}^2\left(x;\lambda_0\right)}{f_\eta^2\left(x;\lambda_0\right)} \leq C\mathbf{1}\left(|x| \leq 1\right) + x^4\frac{f_{\eta,x}^2\left(x;\lambda_0\right)}{f_\eta^2\left(x;\lambda_0\right)}\mathbf{1}\left(|x| > 1\right),$$

where $\mathbf{1}(\cdot)$ signifies the indicator function. Part (iii) follows from parts (i) and (ii) because, due to the Cauchy-Schwarz inequality, $E[e_{x,t}^2\eta_t^2] \leq \left(E[e_{x,t}^2\eta_t^4]E[e_{x,t}^2]\right)^{1/2}$. Similarly, because $E[\left|e_{x,t}\eta_t^3\right|] \leq \left(E[e_{x,t}^2\eta_t^4]E[\eta_t^2]\right)^{1/2}$ part (iv) follows from part (i) and the fact $E[\eta_t^2] = 1$. For part (v) it suffices to show the result $E[e_{\lambda_i,t}^2] < \infty$ for the diagonal elements of the matrix. As

$$E[e_{\lambda_i,t}^2] = \int \frac{f_{\eta,\lambda_i}^2(x;\lambda_0)}{f_\eta^2\left(x;\lambda_0\right)} f_\eta\left(x;\lambda_0\right)dx$$

the finiteness follows from the second dominance condition in Assumption 4(iv). By the Cauchy-Schwarz inequality, $E\left[\left|e_{\lambda_i,t}e_{x,t}\eta_t\right|\right] \leq \left(E[e_{\lambda_i,t}^2]E[e_{x,t}^2\eta_t^2]\right)^{1/2}$ for each component, and hence part (vi) follows from parts (iii) and (v). As $e_{x,t} = f_{\eta,x}\left(\eta_t;\lambda_0\right)/f_\eta\left(\eta_t;\lambda_0\right)$, where the denominator contains a symmetric function and the numerator an odd function, part (vii) follows from Lemma A.3 and the fact $E[|e_{x,t}|] < \infty$ obtained from part (ii). Part (viii) follows because, due to facts $E[\eta_t] = 0$ and $\int f_\eta(x;\lambda_0)dx = 1$,

$$\int x f_{\eta,x}(x;\lambda_0)dx = |_{-\infty}^{\infty} x f_\eta(x;\lambda_0) - \int f_\eta(x;\lambda_0)dx = -1.$$

As in part (vii), $e_{x,t}\eta_t^2$ is an odd function of η_t, and hence part (ix) follows from Lemma A.3 and the fact $E[|e_{x,t}|\,\eta_t^2] < \infty$ obtained from part (i). For part (x), integration by parts yields

$$E\left[e_{x,t}\eta_t^3\right] = \int x^3 f_{\eta,x}(x;\lambda_0)\,dx = |_{-\infty}^{\infty} x^3 f_\eta(x;\lambda_0) - 3\int x^2 f_\eta(x;\lambda_0)\,dx = -3,$$

where the last equality follows because $E[|\eta_t|^3] < \infty$ and $E\left[\eta_t^2\right] = 1$. Part (xi) holds because

$$E\left[e_{\lambda,t}\right] = \int f_{\eta,\lambda}(x;\lambda_0)dx = \int \left[\frac{\partial f_\eta(x;\lambda)}{\partial\lambda}\right]_{\lambda=\lambda_0} dx = \left[\frac{d}{d\lambda}\int f_\eta(x;\lambda)dx\right]_{\lambda=\lambda_0} = 0,$$

where the second last equality is justified by Assumption 4(v) and a theorem justifying the differentiation of an integral (see, e.g., Theorem 24.5 and the discussion following it in Aliprantis and Burkinshaw (1998)). The justification of part (xii) is almost identical, again relying on Assumption 4(v), and also on Assumption 4(ii) to ensure that the last equality holds. ■

Proof of Lemma 2, remaining details. Here we provide several additional details to the proof of Lemma 2.

Remaining details for Step 2(i). To show that $E\left[l_{d,t}(\theta_0)l'_{a,t}(\theta_0)\right] = 0$, write

$$\begin{aligned} E\left[l_{d,t}(\theta_0)l'_{a,t}(\theta_0)\right] &= E\left[e_{x,t}e_{\lambda,t}\frac{u'_{a,t}(\theta_0)}{\sigma_t} - \frac{1}{2}\left(e_{x,t}\eta_t + 1\right)e_{\lambda,t}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2}\right] \\ &= E\left[e_{x,t}e_{\lambda,t}\right]E\left[\frac{u'_{a,t}(\theta_0)}{\sigma_t}\right] - \frac{1}{2}E\left[\left(e_{x,t}\eta_t + 1\right)e_{\lambda,t}\right]E\left[\frac{h'_{a,t}(\theta_0)}{\sigma_t^2}\right], \end{aligned}$$

where the latter equality holds because $h_{a,t}(\theta_0)$, $u_{a,t}(\theta_0)$, and σ_t are $\mathcal{F}_{t-1}^\eta$–measurable. Using the expansions of $u_{a_p,t}(\theta_0)$ and $h_{a_p,t}(\theta_0)$ (see (12a) and (12c)) and the fact that σ_t is an even function of η_τ for all τ it can be seen that both $\sigma_t^{-1}u'_{a,t}(\theta_0)$ and $\sigma_t^{-2}h'_{a,t}(\theta_0)$ can be expressed as sums of components each of which is an odd function of a particular η_τ. Thus, by Lemma A.3, the corresponding expectations are zero, and hence $E\left[l_{d,t}(\theta_0)l'_{a,t}(\theta_0)\right] = 0$.

For the second term of $l_{c,t}(\theta_0)l'_{b,s}(\theta_0)$ and for the two terms of $l_{d,t}(\theta_0)l'_{b,s}(\theta_0)$, one can use the expansions of $u_{b_q,s}(\theta_0)$ and $h_{b_q,s}(\theta_0)$ (see (12b) and (12d)) and similarly as with the first term of $l_{c,t}(\theta_0)l'_{b,s}(\theta_0)$ conclude that it suffices to show that the expressions

$$\frac{h_{c,t}(\theta_0)\sigma_{s-r}\sigma_{s-r+q+j}}{\sigma_t^2\sigma_s^2}\eta_{s-r}\eta_{s-r+q+j}\left(e_{x,t}\eta_t + 1\right)\left(e_{x,s}\eta_s + 1\right)$$

$$\frac{\sigma_{s+q+j}}{\sigma_s}\eta_{s+q+j}e_{x,s}e_{\lambda,t}$$

$$\frac{\sigma_{s-r}\sigma_{s-r+q+j}}{\sigma_s^2}\eta_{s-r}\eta_{s-r+q+j}\left(e_{x,s}\eta_s + 1\right)e_{\lambda,t}$$

have expectation zero. If $t = s$, this is clear for all three as then they are odd functions of the first η appearing in them (see Lemma A.3). Thus, consider the case $t \neq s$.

As for the first expression, if $s > t$, then the largest index involved is either s or $s - r + q + j$. If $s \geq s - r + q + j$, the expectation is zero because both $e_{x,s}\eta_s + 1$ and $\eta_s\left(e_{x,s}\eta_s + 1\right)$ are independent of the remaining terms and have expectation zero. If $s < s - r + q + j$, the expression is an odd function of $\eta_{s-r+q+j}$, implying zero expectation. On the other hand, if $s < t$, the largest index is either t or $s - r + q + j$, and the same argument as in the case $s > t$ shows that the expectation is zero.

Now consider the second expression. If $t \neq s + q + j$, the expression is an odd function of η_{s+q+j} and, if $t = s + q + j \neq s$, it is an odd function of η_s (because $e_{x,s}$ is an odd function of η_s). Hence, it follows that the expectation is always zero.

Finally, consider the third expression. If $s < t$, this expression is an odd function of η_{s-r} and, consequently, has zero expectation. If $s > t$, the largest index involved is either s or $s - r + q + j$ and the argument used for the first expression, case $s > t$, shows that the expectation is zero.

Remaining details for Step 2(iii). We need to establish the expressions of $\mathbf{A}_{21}$ and $\mathbf{B}_{21}$ in Lemma 2. To this end, write $l_{b,t}(\theta_0)l'_{a,s}(\theta_0)$ as

$$\begin{aligned} l_{b,t}(\theta_0)l'_{a,s}(\theta_0) &= e_{x,t}e_{x,s}\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u'_{a,s}(\theta_0)}{\sigma_s} + \frac{1}{4}\left(e_{x,t}\eta_t + 1\right)\left(e_{x,s}\eta_s + 1\right)\frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{h'_{a,s}(\theta_0)}{\sigma_s^2} \\ &\quad -\frac{1}{2}e_{x,t}\left(e_{x,s}\eta_s + 1\right)\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{h'_{a,s}(\theta_0)}{\sigma_s^2} - \frac{1}{2}e_{x,s}\left(e_{x,t}\eta_t + 1\right)\frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{u'_{a,s}(\theta_0)}{\sigma_s}. \end{aligned}$$

We begin with the three last terms of $l_{b,t}(\theta_0)l'_{a,s}(\theta_0)$ because it will turn out that these have zero expectation whenever $t \neq s$. We first establish this. Using the expansions of $u_{a_p,t}(\theta_0)$, $u_{b_q,t}(\theta_0)$, $h_{a_p,t}(\theta_0)$, and $h_{b_q,t}(\theta_0)$ in (12a)–(12d) we can conclude as in Step 2(i) that it suffices to show that the three expressions

$$\begin{aligned} &\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{s-\tilde{r}}\sigma_{s-\tilde{r}-p-i}}{\sigma_t^2\sigma_s^2}\eta_{t-r}\eta_{t-r+q+j}\eta_{s-\tilde{r}}\eta_{s-\tilde{r}-p-i}\left(e_{x,t}\eta_t + 1\right)\left(e_{x,s}\eta_s + 1\right) \\ &\frac{\sigma_{t+q+j}\sigma_{s-r}\sigma_{s-r-p-i}}{\sigma_t\sigma_s^2}\eta_{t+q+j}\eta_{s-r}\eta_{s-r-p-i}e_{x,t}\left(e_{x,s}\eta_s + 1\right) \\ &\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{s-p-i}}{\sigma_t^2\sigma_s}\eta_{t-r}\eta_{t-r+q+j}\eta_{s-p-i}e_{x,s}\left(e_{x,t}\eta_t + 1\right) \end{aligned}$$

have expectation zero whenever $t \neq s$.

Concerning the first expression, if $t > s$, the largest index therein is either t or $t - r + q + j$. In the three possible cases (with relations $<$, $=$, and $>$ between the two indices), either $\eta_{t-r+q+j}$, $\eta_t\left(e_{x,t}\eta_t + 1\right)$, or $e_{x,t}\eta_t + 1$ has expectation zero and is independent of the other terms in the expression. Hence the expectation is zero. If $t < s$, the largest index is either s or $t - r + q + j$, and same reasoning as in the case $t > s$ can be used to deduce that the expectation is zero.

Now consider the second expression. The largest index is always either $t + q + j$ or s, and in the same way as in the previous paragraph, the expectation will always be zero (even in the case $t = s$).

In the third expression, the case $t > s$ can be handled as in the case of the first expression because now the largest index is either t or $t-r+q+j$. The case $t < s$ requires more attention. If $s \neq t-r+q+j$, one of these indices is the largest and the previous argument applies. Therefore assume $s = t-r+q+j$. If $t - r \neq s - p - i$, the considered expression will necessarily be an odd function of either η_{t-r} or η_{s-p-i}, and the expectation will be zero. Thus, assume that also $t - r = s - p - i$ holds. With these restrictions, the considered expression simplifies to

$$\frac{\sigma_{t-r}^2}{\sigma_t^2} \eta_{t-r}^2 \eta_s e_{x,s} \left(e_{x,t} \eta_t + 1 \right).$$

Here the term $e_{x,t}\eta_t + 1$ has expectation zero and is independent of the other terms, implying that the whole expression has zero expectation (note that a crucial point in this argument is that the terms $\sigma_{t-r+q+j}$ and σ_s cancel out and that σ_{s-p-i} $(= \sigma_{t-r})$ is independent of $e_{x,t}\eta_t + 1$).

The second and fourth terms of $l_{b,t}(\theta_0) l'_{a,s}(\theta_0)$ have non-zero expectation when $t = s$, and these non-zero expectations yield the expression of $\mathbf{A}_{21}$ in Lemma 2.

Finally, consider the first term in $l_{b,t}(\theta_0) l'_{a,s}(\theta_0)$, whose element (q, p) is given by

$$e_{x,t} e_{x,s} \frac{u_{b_q,t}(\theta_0)}{\sigma_t} \frac{u'_{a_p,s}(\theta_0)}{\sigma_s} = -\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \psi_{0,j}^{(b)} \psi_{0,i}^{(a)} \frac{\sigma_{t+q+j} \sigma_{s-p-i}}{\sigma_t \sigma_s} e_{x,t} e_{x,s} \eta_{t+q+j} \eta_{s-p-i},$$

where the equality follows from (12a) and (12b). The largest time index in this expression is always either s or $t+q+j$, and to avoid an odd expression as a function of either η_s or η_{t+q+j}, we must have $s = t+q+j$. Then, we must also have $t = s-p-i$ and, necessarily, also $t < s$ and $s-t = p+i = q+j$. With these restrictions all the σ's cancel out and, moreover, $E[e_{x,t} e_{x,s} \eta_{t+q+j} \eta_{s-p-i}] = E[e_{x,t}\eta_t] E[e_{x,s}\eta_s] = 1$ (see Lemma C.1(viii)). It thus suffices to consider the expression

$$-\sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \psi_{0,j}^{(b)} \psi_{0,i}^{(a)}$$

under the restrictions listed above. When $q \geq p$, we may solve for i as $i = j + q - p$, yielding the expression $-\sum_{j=0}^{\infty} \psi_{0,j}^{(b)} \psi_{0,j+q-p}^{(a)} = -\sum_{j=0}^{\infty} \psi_{0,j-q}^{(b)} \psi_{0,j-p}^{(a)}$ when $\psi_{0,j}^{(b)} = \psi_{0,j}^{(a)} = 0$, $j < 0$, is assumed. Thus, we have obtained the expression of $(\mathbf{B}_{21})_{q,p}$ given in Lemma 2.

Remaining details for Step 3. The other three expectations appearing in the expressions of $\mathbf{A}_{21}$ and $\mathbf{A}_{22}$ can be handled in a similar manner. Making use of the expansions (12a) and (12d), the element (q, p) of the matrix $h_{b,t}(\theta_0) u'_{a,t}(\theta_0) e_{x,t} \left(e_{x,t}\eta_t + 1 \right) / \sigma_t^3$ can be written as

$$-2 \sum_{r=1}^{R} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_{0,r} \psi_{0,j}^{(b)} \psi_{0,i}^{(a)} \frac{\sigma_{t-r} \sigma_{t-r+q+j} \sigma_{t-p-i}}{\sigma_t^3} \eta_{t-r} \eta_{t-r+q+j} \eta_{t-p-i} e_{x,t} \left(e_{x,t}\eta_t + 1 \right).$$

In this summation, only the terms in which $t - r = t - p - i$ and $t = t - r + q + j$ have nonzero expectation, and thus it suffices to show that $\sigma_{t-r}^2 \eta_{t-r}^2 \eta_t e_{x,t} \left(e_{x,t}\eta_t + 1 \right) / \sigma_t^2$ has finite expectation. This follows from the assumptions and arguments already used.

The element $(q, \tilde{q})$ of the matrix $u_{b,t}(\theta_0)u'_{b,t}(\theta_0)e^2_{x,t}/\sigma^2_t$ can be written as (see (12b))

$$\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\psi^{(b)}_{0,j}\psi^{(b)}_{0,\tilde{j}}\frac{\sigma_{t+q+j}\sigma_{t+\tilde{q}+\tilde{j}}}{\sigma^2_t}\eta_{t+q+j}\eta_{t+\tilde{q}+\tilde{j}}e^2_{x,t}.$$

For nonzero expectation we must have $q+j=\tilde{q}+\tilde{j}$, and the resulting expression has finite expectation by Lemma B.1 and Lemma C.1.

Finally, consider the element $(q, \tilde{q})$ of the matrix $h_{b,t}(\theta_0)h'_{b,t}(\theta_0)\left(e_{x,t}\eta_t+1\right)^2/\sigma^4_t$, which can be written as (see (12d))

$$4\sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi^{(b)}_{0,j}\psi^{(b)}_{0,\tilde{j}}\frac{\sigma_{t-r}\sigma_{t-\tilde{r}}\sigma_{t-r+q+j}\sigma_{t-\tilde{r}+\tilde{q}+\tilde{j}}}{\sigma^4_t}\eta_{t-r}\eta_{t-r+q+j}\eta_{t-\tilde{r}}\eta_{t-\tilde{r}+\tilde{q}+\tilde{j}}\left(e_{x,t}\eta_t+1\right)^2.$$

Only the terms in the summation with $r=\tilde{r}$ and $q+j=\tilde{q}+\tilde{j}$ have nonzero expectations, and thus it suffices to consider the expression

$$\frac{\sigma^2_{t-r}\sigma^2_{t-r+q+j}}{\sigma^4_t}\eta^2_{t-r}\eta^2_{t-r+q+j}\left(e_{x,t}\eta_t+1\right)^2.$$

If $-r+q+j<0$, this is the same expression as already considered in the case of the first term. On the other hand, if $-r+q+j\geq 0$ the arguments used therein in conjunction with Lemma B.1 and Lemma C.1 give the result.

The result $\text{Cov}\left[x_t\right]=\mathcal{I}_1\left(\theta_0\right)$ **in Step 4.** To show that $\text{Cov}\left[x_t\right]=\mathcal{I}_1\left(\theta_0\right)$, denote

$$\mathbf{A}_{11,1}=E\left[e^2_{x,t}\right]E\left[\tfrac{u_{a,t}(\theta_0)}{\sigma_t}\tfrac{u'_{a,t}(\theta_0)}{\sigma_t}\right],\qquad \mathbf{A}_{11,2}=\tfrac{1}{4}E\left[\left(e_{x,t}\eta_t+1\right)^2\right]E\left[\tfrac{h_{a,t}(\theta_0)}{\sigma^2_t}\tfrac{h'_{a,t}(\theta_0)}{\sigma^2_t}\right],$$

$$\mathbf{A}_{21,1}=\tfrac{1}{4}E\left[\left(e_{x,t}\eta_t+1\right)^2\tfrac{h_{b,t}(\theta_0)}{\sigma^2_t}\tfrac{h'_{a,t}(\theta_0)}{\sigma^2_t}\right]\quad \mathbf{A}_{21,2}=-\tfrac{1}{2}E\left[e_{x,t}\left(e_{x,t}\eta_t+1\right)\tfrac{h_{b,t}(\theta_0)}{\sigma^2_t}\tfrac{u'_{a,t}(\theta_0)}{\sigma_t}\right],$$

$$\mathbf{A}_{22,1}=E\left[e^2_{x,t}\tfrac{u_{b,t}(\theta_0)}{\sigma_t}\tfrac{u'_{b,t}(\theta_0)}{\sigma_t}\right],\qquad \mathbf{A}_{22,2}=\tfrac{1}{4}E\left[\left(e_{x,t}\eta_t+1\right)^2\tfrac{h_{b,t}(\theta_0)}{\sigma^2_t}\tfrac{h'_{b,t}(\theta_0)}{\sigma^2_t}\right].$$

From the definitions it follows that $x_{a,t}=x_{a,1,t}+x_{a,2,t}=l_{a,t}\left(\theta_0\right)$ and that

$$\text{Cov}\left[x_{a,t}\right]=\text{Cov}\left[x_{a,1,t}\right]+\text{Cov}\left[x_{a,2,t}\right]=\mathbf{A}_{11,1}+\mathbf{A}_{11,2}=\mathbf{A}_{11}.$$

Concerning the covariance between $x_{b,t}$ and $x_{a,t}$, first note that $\text{Cov}[x_{b_q,2,t},x_{a_p,1,t}]=0$ (because each term in $x_{b_q,2,t}x_{a_p,1,t}$ will necessarily be an odd function of either η_{t-r} or $\eta_{t-r+q+j}$), $\text{Cov}[x_{b_q,1,t},x_{a_p,2,t}]=0$ (each term is necessarily an odd function of either η_{t-r} or $\eta_{t-r-p-i}$), and $\text{Cov}[x_{b_q,3,t},x_{a_p,2,t}]=0$ (if $q+j>r$, the term $\left(e_{x,t}\eta_t+1\right)\eta_t$ will be independent of the rest with expectation zero; if $q+j=r$, each term is necessarily an odd function of either η_{t-r} or $\eta_{t-r-p-i}$). Moreover,

$$\text{Cov}\left[x_{b_q,1,t},x_{a_p,1,t}\right]=-\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\psi^{(a)}_{0,i}\psi^{(b)}_{0,j}E\left[e_{x,t-q-j}\eta_{t-p-i}\right]E\left[e_{x,t}\eta_t\right]\mathbf{1}\left(q+j=i+p\right)=-\sum_{j=0}^{\infty}\psi^{(b)}_{0,j-q}\psi^{(a)}_{0,j-p},$$

so that $\text{Cov}\left[x_{b,1,t}, x_{a,1,t}\right] = \mathbf{B}_{21}$.

The covariance between $x_{b_q,3,t}$ and $x_{a_p,1,t}$ is obtained as the expected value of

$$x_{b_q,3,t}x_{a_p,1,t} = \sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,i}^{(a)}\frac{\sigma_{t-q-j}\sigma_{t-p-i}}{\sigma_{t+r-q-j}^2}\eta_{t-q-j}\eta_{t-p-i}\left(e_{x,t+r-q-j}\eta_{t+r-q-j}+1\right) \times e_{x,t}\eta_t\mathbf{1}\left(q+j\geq r\right)$$

and a non-zero expectation can only be obtained when $r = q + j = p + i$. For this covariance, it thus suffices to consider the expected value of

$$\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,i}^{(a)}\frac{\sigma_{t-r}^2}{\sigma_t^2}\eta_{t-r}^2\left(e_{x,t}\eta_t+1\right)e_{x,t}\eta_t\mathbf{1}\left(r=q+j=p+i\right).$$

On the other hand, from the definition of the matrix $\mathbf{A}_{21,2}$ (see also (12a) and (12d)) it can be seen that its elements are defined as the expectations of the variables

$$\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,i}^{(a)}\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{t-p-i}}{\sigma_t^2\sigma_t}\eta_{t-r}\eta_{t-r+q+j}\eta_{t-p-i}e_{x,t}\left(e_{x,t}\eta_t+1\right),$$

and non-zero expectations are obtained only when $r = q + j = p + i$. Thus $\text{Cov}\left[x_{b,3,t}x_{a,1,t}\right] = \mathbf{A}_{21,2}$.

Finally, the covariance between $x_{b_q,2,t}$ and $x_{a_p,2,t}$ is obtained as the expected value of

$$x_{b_q,2,t}x_{a_p,2,t} = -\sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,i}^{(a)}\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{t-\tilde{r}}\sigma_{t-\tilde{r}-p-i}}{\sigma_t^2\sigma_t^2}\eta_{t-r}\eta_{t-r+q+j}\eta_{t-\tilde{r}}\eta_{t-\tilde{r}-p-i} \times \left(e_{x,t}\eta_t+1\right)^2\mathbf{1}\left(q+j<r\right).$$

The only terms in the summation with non-zero expectation are those in which $r = \tilde{r} + p + i$ and $r = \tilde{r} + q + j$, and thus also $p + i = q + j$. For this covariance, it thus suffices to consider the expected value of

$$-\sum_{\tilde{r}=1}^{R}\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,i}^{(a)}\frac{\sigma_{t-r}^2\sigma_{t-r+q+j}^2}{\sigma_t^2\sigma_t^2}\eta_{t-r}^2\eta_{t-r+q+j}^2\left(e_{x,t}\eta_t+1\right)^2 \times \mathbf{1}\left(r=\tilde{r}+p+i=\tilde{r}+q+j, q+j<r\right).$$

On the other hand, from the definition of the matrix $\mathbf{A}_{21,1}$ (see also (12c) and (12d)) it can be seen that its elements are defined as the expectations of the variables

$$-\sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,i}^{(a)}\psi_{0,j}^{(b)}\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{t-\tilde{r}}\sigma_{t-\tilde{r}-p-i}}{\sigma_t^2\sigma_t^2}\eta_{t-r}\eta_{t-r+q+j}\eta_{t-\tilde{r}}\eta_{t-\tilde{r}-p-i}\left(e_{x,t}\eta_t+1\right)^2,$$

and non-zero expectations are obtained only when $r = \tilde{r} + p + i = \tilde{r} + q + j$, $q + j < r$, so that $\text{Cov}\left[x_{b,2,t}, x_{a,2,t}\right] = \mathbf{A}_{21,1}$. Combining the results above, we find that $\text{Cov}\left[x_{b,t}, x_{a,t}\right] = \mathbf{A}_{21} + \mathbf{B}_{21}$.

What remains is to show that $\text{Cov}\left[x_{b,t}\right] = \mathbf{A}_{22} + \mathbf{B}_{22}$. We begin by showing that the covariances $\text{Cov}\left[x_{b,1,t}, x_{b,2,t}\right]$ and $\text{Cov}\left[x_{b,2,t}, x_{b,3,t}\right]$ are zero matrices. The elements of the former are obtained as the expected values of

$$x_{b_q,1,t}x_{b_{\tilde{q}},2,t} = -\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_t\sigma_{t-\tilde{r}}\sigma_{t-\tilde{r}+\tilde{q}+\tilde{j}}}{\sigma_{t-q-j}\sigma_t^2}e_{x,t-q-j}\eta_{t-\tilde{r}}\eta_{t-\tilde{r}+\tilde{q}+\tilde{j}}\left(e_{x,t}\eta_t+1\right)\eta_t\mathbf{1}\left(\tilde{q}+\tilde{j}<\tilde{r}\right)$$

and non-zero expectations could only be obtained if $\tilde{r} = q + j = \tilde{q} + \tilde{j}$, which is not possible given the restriction $\tilde{q} + \tilde{j} < \tilde{r}$. Elements of the latter are obtained as the expected values of

$$\begin{aligned} x_{b_q,2,t}x_{b_{\tilde{q}},3,t} &= \sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{t-\tilde{q}-\tilde{j}}\sigma_t}{\sigma_{t+\tilde{r}-\tilde{q}-\tilde{j}}^2\sigma_t^2}\eta_{t-r}\eta_{t-r+q+j}\eta_{t-\tilde{q}-\tilde{j}} \\ &\quad\times\left(e_{x,t+\tilde{r}-\tilde{q}-\tilde{j}}\eta_{t+\tilde{r}-\tilde{q}-\tilde{j}}+1\right)\left(e_{x,t}\eta_t+1\right)\eta_t\mathbf{1}\left(q+j<r\right)\mathbf{1}\left(\tilde{q}+\tilde{j}\geq\tilde{r}\right). \end{aligned}$$

As $\left(e_{x,t}\eta_t+1\right)\eta_t$ has zero expectation, non-zero expectation could only be obtained if $\tilde{r} = \tilde{q} + \tilde{j}$, but in this case the expression will necessarily be an odd function of either η_{t-r} or $\eta_{t-r+q+j}$.

Now consider the covariance $\text{Cov}\left[x_{b,3,t}, x_{b,1,t}\right]$, whose elements are obtained as the expected values of the terms

$$\begin{aligned} x_{b_q,3,t}x_{b_{\tilde{q}},1,t} &= -\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t-q-j}\sigma_t^2}{\sigma_{t-\tilde{q}-\tilde{j}}\sigma_{t+r-q-j}^2}\eta_{t-q-j}e_{x,t-\tilde{q}-\tilde{j}}\left(e_{x,t+r-q-j}\eta_{t+r-q-j}+1\right)\eta_t^2 \\ &\quad\times\mathbf{1}\left(q+j\geq r\right). \end{aligned}$$

The restriction $q + j = \tilde{q} + \tilde{j}$ must hold, and under this restriction the expression simplifies to

$$\begin{aligned} x_{b_q,3,t}x_{b_{\tilde{q}},1,t} &= -\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_t^2}{\sigma_{t+r-q-j}^2}\eta_{t-q-j}e_{x,t-q-j}\left(e_{x,t+r-q-j}\eta_{t+r-q-j}+1\right)\eta_t^2 \\ &\quad\times\mathbf{1}\left(\tilde{q}+\tilde{j}=q+j\geq r\right). \end{aligned}$$

Due to stationarity, the above expression has the same expected value as (replace t with $t+q+j-r$)

$$-\sum_{r=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,j+q-\tilde{q}}^{(b)}\frac{\sigma_{t+q+j-r}^2}{\sigma_t^2}\eta_{t-r}e_{x,t-r}\left(e_{x,t}\eta_t+1\right)\eta_{t+q+j-r}^2\mathbf{1}\left(q+j-r\geq 0\right).$$

In the case $q + j - r = 0$, the random part of the above expression has expectation equal to 2 ($E\left[\eta_{t-r}e_{x,t-r}\right] = -1$ and $E\left[\left(e_{x,t}\eta_t+1\right)\eta_t^2\right] = -3 + 1 = -2$), whereas in the case $q + j - r > 0$, the expectation of the random part equals $2c_{q+j-r}$ by Lemma B.1 ($H_{t-1} = \eta_{t-r}e_{x,t-r}$, $g(\eta_t) = e_{x,t}\eta_t + 1$). Therefore, as $c_0 = 1$,

$$E\left[x_{b_q,3,t}x_{b_{\tilde{q}},1,t}\right] = -2\sum_{r=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,j+q-\tilde{q}}^{(b)}c_{q+j-r}\mathbf{1}\left(q+j-r\geq 0\right)$$

so that, making use of the conventions $\psi_{0,j}^{(b)} = 0$ for $j < 0$ and $c_a = 0$ for $a < 0$,

$$\mathrm{Cov}\left[x_{b_q,3,t}, x_{b_{\tilde{q}},1,t}\right] = -2\sum_{r=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,r}\psi_{0,j-q}^{(b)}\psi_{0,j-\tilde{q}}^{(b)}c_{j-r}.$$

As this equals half of the element $(q,\tilde{q})$ of $\mathbf{B}_{22}$, we conclude that $\mathrm{Cov}\left[x_{b,3,t}, x_{b,1,t}\right] + \mathrm{Cov}\left[x_{b,1,t}, x_{b,3,t}\right] = \mathbf{B}_{22}$.

What remains is to obtain the matrix $\mathbf{A}_{22}$ from the covariances $\mathrm{Cov}\left[x_{b,1,t}\right]$, $\mathrm{Cov}\left[x_{b,2,t}\right]$, and $\mathrm{Cov}\left[x_{b,3,t}\right]$. Concerning the first one,

$$E\left[x_{b_q,1,t}x_{b_{\tilde{q}},1,t}\right] = \sum_{j=0}^{\infty}\psi_{0,j-q}^{(b)}\psi_{0,j-\tilde{q}}^{(b)}E\left[\frac{\sigma_t^2}{\sigma_{t-q-j}^2}e_{x,t-q-j}^2\right] = \sum_{j=0}^{\infty}\psi_{0,j-q}^{(b)}\psi_{0,j-\tilde{q}}^{(b)}E\left[\frac{\sigma_{t+q+j}^2}{\sigma_t^2}e_{x,t}^2\right],$$

where the first equality is obtained by noting that $q + j = \tilde{q} + \tilde{j}$ must hold, and second is due to stationarity. The last expression equals the element $(q,\tilde{q})$ of $\mathbf{A}_{22,1}$, and thus $\mathrm{Cov}\left[x_{b,1,t}\right] = \mathbf{A}_{22,1}$.

The covariance between $x_{b_q,2,t}$ and $x_{b_{\tilde{q}},2,t}$ equals the expected value of

$$\begin{aligned} x_{b_q,2,t}x_{b_{\tilde{q}},2,t} &= \sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t-r}\sigma_{t-\tilde{r}}\sigma_{t-r+q+j}\sigma_{t-\tilde{r}+\tilde{q}+\tilde{j}}}{\sigma_t^2\sigma_t^2}\eta_{t-r}\eta_{t-\tilde{r}}\eta_{t-r+q+j}\eta_{t-\tilde{r}+\tilde{q}+\tilde{j}} \\ &\quad \times \left(e_{x,t}\eta_t + 1\right)^2 \mathbf{1}\left(q+j<r\right)\mathbf{1}\left(\tilde{q}+\tilde{j}<\tilde{r}\right), \end{aligned}$$

and non-zero expectation only results from summands with $r = \tilde{r}$ and $q + j = \tilde{q} + \tilde{j}$. Therefore,

$$\mathrm{Cov}\left[x_{b_q,2,t}, x_{b_{\tilde{q}},2,t}\right] = \sum_{r=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,r}^2\psi_{0,j}^{(b)}\psi_{0,j+q-\tilde{q}}^{(b)}E\left[\frac{\sigma_{t-r}^2\sigma_{t-r+q+j}^2}{\sigma_t^2\sigma_t^2}\eta_{t-r}^2\eta_{t-r+q+j}^2\left(e_{x,t}\eta_t+1\right)^2\right]\mathbf{1}\left(q+j<r\right),$$

which almost equals the element $(q,\tilde{q})$ of $\mathbf{A}_{22,2}$, the difference being the multiplication by the indicator function $\mathbf{1}\left(q+j<r\right)$.

Finally, consider the covariance between $x_{b_q,3,t}$ and $x_{b_{\tilde{q}},3,t}$, which is obtained as the expected value of

$$\begin{aligned} x_{b_q,3,t}x_{b_{\tilde{q}},3,t} &= \sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_t^2\sigma_{t-q-j}\sigma_{t-\tilde{q}-\tilde{j}}}{\sigma_{t+r-q-j}^2\sigma_{t+\tilde{r}-\tilde{q}-\tilde{j}}^2}\eta_{t-q-j}\eta_{t-\tilde{q}-\tilde{j}} \\ &\quad \times \left(e_{x,t+r-q-j}\eta_{t+r-q-j}+1\right)\left(e_{x,t+\tilde{r}-\tilde{q}-\tilde{j}}\eta_{t+\tilde{r}-\tilde{q}-\tilde{j}}+1\right)\eta_t^2\mathbf{1}\left(q+j\geq r\right)\mathbf{1}\left(\tilde{q}+\tilde{j}\geq\tilde{r}\right). \end{aligned}$$

In order to have non-zero expectation, the restriction $q+j = \tilde{q}+\tilde{j}$ must hold, and under this restriction we have

$$\begin{aligned} x_{b_q,3,t}x_{b_{\tilde{q}},3,t} &= \sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,j+q-\tilde{q}}^{(b)}\frac{\sigma_t^2\sigma_{t-q-j}^2}{\sigma_{t+r-q-j}^2\sigma_{t+\tilde{r}-q-j}^2}\eta_{t-q-j}^2 \\ &\quad \times \left(e_{x,t+r-q-j}\eta_{t+r-q-j}+1\right)\left(e_{x,t+\tilde{r}-q-j}\eta_{t+\tilde{r}-q-j}+1\right)\eta_t^2\mathbf{1}\left(q+j\geq r\right)\mathbf{1}\left(q+j\geq\tilde{r}\right). \end{aligned}$$

By stationarity, the this expression has the same expected value as (replace t with $t+q+j-r$)

$$\sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,j+q-\tilde{q}}^{(b)}\frac{\sigma_{t+q+j-r}^{2}\sigma_{t-r}^{2}}{\sigma_{t}^{2}\sigma_{t-r+\tilde{r}}^{2}}\eta_{t-r}^{2}$$
$$\times\left(e_{x,t}\eta_{t}+1\right)\left(e_{x,t-r+\tilde{r}}\eta_{t-r+\tilde{r}}+1\right)\eta_{t+q+j-r}^{2}\mathbf{1}\left(q+j\geq r\right)\mathbf{1}\left(q+j\geq\tilde{r}\right).$$

We next show that the random part in this expression has expected value zero when $\tilde{r}\neq r$. If $\tilde{r}>r$, this can be seen by applying Lemma B.1 with $g(\eta_{t-r+\tilde{r}})=e_{x,t-r+\tilde{r}}\eta_{t-r+\tilde{r}}+1$ and $H_{t-r+\tilde{r}-1}=\frac{\sigma_{t-r}^{2}}{\sigma_{t}^{2}}\eta_{t-r}^{2}\left(e_{x,t}\eta_{t}+1\right)$. On the other hand, when $\tilde{r}<r$, this can again be seen by Lemma B.1, this time choosing $g(\eta_{t})=e_{x,t}\eta_{t}+1$ and $H_{t-1}=\frac{\sigma_{t-r}^{2}}{\sigma_{t-r+\tilde{r}}^{2}}\eta_{t-r}^{2}(e_{x,t-r+\tilde{r}}\eta_{t-r+\tilde{r}}+1)$. Therefore, non-zero expectation can only be obtained when $\tilde{r}=r$, and thus

$$E\left[x_{b_{q},3,t}x_{b_{\tilde{q}},3,t}\right]=\sum_{r=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,r}^{2}\psi_{0,j}^{(b)}\psi_{0,j+q-\tilde{q}}^{(b)}E\left[\frac{\sigma_{t-r+q+j}^{2}\sigma_{t-r}^{2}}{\sigma_{t}^{2}\sigma_{t}^{2}}\eta_{t-r}^{2}\eta_{t-r+q+j}^{2}\left(e_{x,t}\eta_{t}+1\right)^{2}\right]\mathbf{1}\left(q+j\geq r\right),$$

which almost equals the $(q,\tilde{q})$–element of $\mathbf{A}_{22,2}$, the difference being the multiplication by the indicator function $\mathbf{1}\left(q+j\geq r\right)$.

Putting the above pieces together we have thus shown that

$$\mathrm{Cov}\left[x_{b,1,t}\right]+\mathrm{Cov}\left[x_{b,2,t}\right]+\mathrm{Cov}\left[x_{b,3,t}\right]=\mathbf{A}_{22,1}+\mathbf{A}_{22,2},$$

and hence that $\mathrm{Cov}\left[x_{b,t}\right]=\mathbf{A}_{22}+\mathbf{B}_{22}$. Therefore, $\mathrm{Cov}\left[x_{t}\right]=\mathcal{I}_{1}\left(\theta_{0}\right)$.

Expression of the matrix Ψ in Step 4. The $(P+Q)\times 2K$ matrix Ψ is defined as

$$\Psi=\begin{bmatrix}\Psi^{(a)} & 0\\ 0 & \Psi^{(b)}\end{bmatrix},$$

where

$$\Psi^{(a)}=\begin{bmatrix}1 & \psi_{0,1}^{(a)} & \psi_{0,2}^{(a)} & \psi_{0,3}^{(a)} & \cdots & \cdots & \psi_{0,K-1}^{(a)}\\ 0 & 1 & \psi_{0,1}^{(a)} & \psi_{0,2}^{(a)} & \cdots & \cdots & \psi_{0,K-2}^{(a)}\\ \vdots & & \ddots & \ddots & & & \vdots\\ 0 & \cdots & 0 & 1 & \psi_{0,1}^{(a)} & \cdots & \psi_{0,K-P}^{(a)}\end{bmatrix},\quad \Psi^{(b)}=\begin{bmatrix}1 & \psi_{0,1}^{(b)} & \psi_{0,2}^{(b)} & \psi_{0,3}^{(b)} & \cdots & \cdots & \psi_{0,K-1}^{(b)}\\ 0 & 1 & \psi_{0,1}^{(b)} & \psi_{0,2}^{(b)} & \cdots & \cdots & \psi_{0,K-2}^{(b)}\\ \vdots & & \ddots & \ddots & & & \vdots\\ 0 & \cdots & 0 & 1 & \psi_{0,1}^{(b)} & \cdots & \psi_{0,K-Q}^{(b)}\end{bmatrix}$$

with dimensions $(P\times K)$ and $(Q\times K)$, respectively.

Deriving a contradiction in the cases $2<k\leq R$ at the end of Step 4. We now show how to derive a contradiction in the cases $2<k\leq R$ at the end of Step 4. Supposing $k>2$ and reorganizing, (23) can be written as

$$-\alpha_{0,k-2}(e_{x,t-2}\eta_{t-2}+1)\eta_{t-k}\frac{\sigma_{t-1}^{2}}{\sigma_{t-2}^{2}}+\left(\frac{e_{x,t-k}}{\sigma_{t-k}^{2}}-\sum_{r=1}^{k-3}\alpha_{0,r}(e_{x,t+r-k}\eta_{t+r-k}+1)\frac{\eta_{t-k}}{\sigma_{t+r-k}^{2}}\right)\sigma_{t-1}^{2}+\alpha_{0,k-1}\lambda_{0}\eta_{t-k}=0$$

or as

$$-\alpha_{0,k-2}\eta_{t-k}\frac{\sigma_{t-1}^2}{\sigma_{t-2}^2}(e_{x,t-2}\eta_{t-2}+1)+\mu_{2,t-3}\sigma_{t-1}^2+\nu_{2,t-3}=0$$

with obvious definitions of the $\mathcal{F}_{t-3}^{\eta}$–measurable variables $\mu_{2,t-3}$ and $\nu_{2,t-3}$. Exactly the same steps as above lead to a polynomial equation where the coefficient of η_{t-2}^4 must satisfy

$$\alpha_{0,k-2}\eta_{t-k}\alpha_{0,1}\sigma_{t-2}^2\lambda_0+\mu_{2,t-3}\alpha_{0,1}\sigma_{t-2}^4=0,$$

or after dividing with $\alpha_{0,1}\sigma_{t-2}^2$, substituting in for $\mu_{2,t-3}$, and reorganizing

$$e_{x,t-k}\frac{\sigma_{t-2}^2}{\sigma_{t-k}^2}-\sum_{r=1}^{k-3}\alpha_{0,r}(e_{x,t+r-k}\eta_{t+r-k}+1)\eta_{t-k}\frac{\sigma_{t-2}^2}{\sigma_{t+r-k}^2}+\alpha_{0,k-2}\eta_{t-k}\lambda_0=0.$$

This is again close to equation (21). Continuing in this fashion we can arrive at a contradiction for $k=3$, ..., $k=R-2$, and at last arrive (while assuming $k>R-2$) at

$$e_{x,t-k}\frac{\sigma_{t-R+2}^2}{\sigma_{t-k}^2}-\sum_{r=1}^{k-R+1}\alpha_{0,r}(e_{x,t+r-k}\eta_{t+r-k}+1)\eta_{t-k}\frac{\sigma_{t-R+2}^2}{\sigma_{t+r-k}^2}+\alpha_{0,k-R+2}\eta_{t-k}\lambda_0=0.$$

If $k=R-1$ this becomes

$$e_{x,t-R+1}\frac{\sigma_{t-R+2}^2}{\sigma_{t-R+1}^2}+\alpha_{0,1}\eta_{t-R+1}\lambda_0=0,$$

giving again a contradiction. Finally, suppose that $k>R-1$, that is, $k=R$. Then we need to consider the equation

$$e_{x,t-R}\frac{\sigma_{t-R+2}^2}{\sigma_{t-R}^2}-\sum_{r=1}^{1}\alpha_{0,r}(e_{x,t+r-R}\eta_{t+r-R}+1)\eta_{t-R}\frac{\sigma_{t-R+2}^2}{\sigma_{t+r-R}^2}+\alpha_{0,2}\eta_{t-R}\lambda_0=0,$$

which can be written as

$$-\alpha_{0,1}\eta_{t-R}\frac{\sigma_{t-R+2}^2}{\sigma_{t-R+1}^2}(e_{x,t-R+1}\eta_{t-R+1}+1)+\mu_{R-1,t-R}\sigma_{t-R+2}^2+\nu_{R-1,t-R}=0.$$

Exactly the same steps as previously lead to a fourth order polynomial equation with the coefficient of η_{t-R+1}^4 given by $\alpha_{0,1}\eta_{t-R}\alpha_{0,1}\sigma_{t-R+1}^2\lambda_0+\mu_{R-1,t-R}\alpha_{0,1}\sigma_{t-R+1}^4$. This expression has to be zero so that, dividing with $\alpha_{0,1}\sigma_{t-R+1}^2$ and substituting in for $\mu_{R-1,t-R}$, we get

$$e_{x,t-R}\frac{\sigma_{t-R+1}^2}{\sigma_{t-R}^2}+\alpha_{0,1}\eta_{t-R}\lambda_0=0,$$

which leads to a contradiction in the same way as before. ■

Proof of Lemma 5. We first note that the Hessian evaluated at the true parameter value takes the form

$$
\begin{aligned}
l_{aa,t}(\theta_0) &= e_{xx,t}\frac{u_{a,t}(\theta_0)}{\sigma_t}\frac{u'_{a,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{1,t}\frac{h_{a,t}(\theta_0)}{\sigma_t^2}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2} \\
&\quad + \mathsf{E}_{2,t}\left(\frac{u_{a,t}(\theta_0)}{\sigma_t}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2} + \frac{h_{a,t}(\theta_0)}{\sigma_t^2}\frac{u'_{a,t}(\theta_0)}{\sigma_t}\right) + e_{x,t}\frac{u_{aa,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{3,t}\frac{h_{aa,t}(\theta_0)}{\sigma_t^2}, \\
l_{ba,t}(\theta_0) &= e_{xx,t}\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u'_{a,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{1,t}\frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2} \\
&\quad + \mathsf{E}_{2,t}\left(\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2} + \frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{u'_{a,t}(\theta_0)}{\sigma_t}\right) + e_{x,t}\frac{u_{ba,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{3,t}\frac{h_{ba,t}(\theta_0)}{\sigma_t^2}, \\
l_{bb,t}(\theta_0) &= e_{xx,t}\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u'_{b,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{1,t}\frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{h'_{b,t}(\theta_0)}{\sigma_t^2} \\
&\quad + \mathsf{E}_{2,t}\left(\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{h'_{b,t}(\theta_0)}{\sigma_t^2} + \frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{u'_{b,t}(\theta_0)}{\sigma_t}\right) + e_{x,t}\frac{u_{bb,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{3,t}\frac{h_{bb,t}(\theta_0)}{\sigma_t^2}, \\
l_{ca,t}(\theta_0) &= \mathsf{E}_{1,t}\frac{h_{c,t}(\theta_0)}{\sigma_t^2}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2} + \mathsf{E}_{2,t}\frac{h_{c,t}(\theta_0)}{\sigma_t^2}\frac{u'_{a,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{3,t}\frac{h_{ca,t}(\theta_0)}{\sigma_t^2}, \\
l_{cb,t}(\theta_0) &= \mathsf{E}_{1,t}\frac{h_{c,t}(\theta_0)}{\sigma_t^2}\frac{h'_{b,t}(\theta_0)}{\sigma_t^2} + \mathsf{E}_{2,t}\frac{h_{c,t}(\theta_0)}{\sigma_t^2}\frac{u'_{b,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{3,t}\frac{h_{cb,t}(\theta_0)}{\sigma_t^2}, \\
l_{cc,t}(\theta_0) &= \mathsf{E}_{1,t}\frac{h_{c,t}(\theta_0)}{\sigma_t^2}\frac{h'_{c,t}(\theta_0)}{\sigma_t^2} + \mathsf{E}_{3,t}\frac{h_{cc,t}(\theta_0)}{\sigma_t^2}, \\
l_{da,t}(\theta_0) &= e_{\lambda x,t}\left(\frac{u'_{a,t}(\theta_0)}{\sigma_t} - \frac{1}{2}\eta_t\frac{h'_{a,t}(\theta_0)}{\sigma_t^2}\right), \qquad l_{db,t}(\theta_0) = e_{\lambda x,t}\left(\frac{u'_{b,t}(\theta_0)}{\sigma_t} - \frac{1}{2}\eta_t\frac{h'_{b,t}(\theta_0)}{\sigma_t^2}\right), \\
l_{dc,t}(\theta_0) &= e_{\lambda x,t}\left(-\frac{1}{2}\eta_t\frac{h'_{c,t}(\theta_0)}{\sigma_t^2}\right), \qquad\qquad l_{dd,t}(\theta_0) = e_{\lambda\lambda,t},
\end{aligned}
$$

where $\mathsf{E}_{1,t}(\theta)$, $\mathsf{E}_{2,t}(\theta)$, and $\mathsf{E}_{3,t}(\theta)$ evaluated at θ_0 are given by

$$
\mathsf{E}_{1,t} = \frac{1}{2}\left(\frac{1}{2}e_{xx,t}\eta_t^2 + \frac{3}{2}e_{x,t}\eta_t + 1\right), \quad \mathsf{E}_{2,t} = -\frac{1}{2}(e_{xx,t}\eta_t + e_{x,t}), \quad \mathsf{E}_{3,t} = -\frac{1}{2}(e_{x,t}\eta_t + 1),
$$

and where

$$
\begin{aligned}
e_{xx,t} &= \frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} - \left(\frac{f_{\eta,x}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\right)^2 = \frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} - e_{x,t}^2 \\
e_{\lambda x,t} &= \frac{f_{\eta,\lambda x}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} - \frac{f_{\eta,\lambda}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\frac{f_{\eta,x}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} = \frac{f_{\eta,\lambda x}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} - e_{\lambda,t}e_{x,t} \\
e_{\lambda\lambda,t} &= \frac{f_{\eta,\lambda\lambda}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} - \frac{f_{\eta,\lambda}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\frac{f'_{\eta,\lambda}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} = \frac{f_{\eta,\lambda\lambda}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} - e_{\lambda,t}e'_{\lambda,t}.
\end{aligned}
$$

The notation

$$
\mathsf{E}_{4,t} = \frac{1}{4}\frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t^2 + \frac{5}{4}e_{x,t}\eta_t + \frac{3}{4}
$$

will also be useful.

We next present a small lemma containing results that will be used to prove Lemma 5. Its proof is given below after the proof of Lemma 5.

Lemma D.1. *If Assumptions 1–5 hold, then (i)* $E[e_{\lambda,t}] = 0$*, (ii)* $E\left[\frac{f_{\eta,\lambda\lambda}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\right] = 0$*, (iii)* $E[\eta_t e_{\lambda,t}] = 0$*, (iv)* $E\left[\eta_t \frac{f_{\eta,\lambda x}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\right] = 0$*, (v)* $E[e_{xx,t}] = -E[e_{x,t}^2]$*, (vi)* $E[e_{xx,t}\eta_t] = 0$*, (vii)* $E[\mathsf{E}_{1,t}] = \frac{1}{4}\left(1 - E\left[e_{x,t}^2\eta_t^2\right]\right)$*, (viii)* $E[\mathsf{E}_{2,t}] = 0$*, (ix)* $E[\mathsf{E}_{3,t}] = 0$*, (x)* $E[\eta_t\mathsf{E}_{1,t}] = 0$*, (xi)* $E[\eta_t\mathsf{E}_{2,t}] = -\frac{1}{2}(1 - E[e_{x,t}^2\eta_t^2])$*, (xii)* $E[\eta_t\mathsf{E}_{3,t}] = 0$*, (xiii)* $E\left[\eta_t^2\mathsf{E}_{3,t}\right] = 1$*, (xiv)* $E[\mathsf{E}_{4,t}] = 0$*, (xv)* $E\left[\frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t^4\right] = 12$*, and (xvi)* $E\left[\eta_t^2\mathsf{E}_{4,t}\right] = 0$.

We are now ready to prove Lemma 5. We begin with the four blocks in the lower left-hand corner (Part i), then consider the four blocks in the lower right-hand corner (Part ii), and finish with the four blocks in the upper left-hand corner (Part iii).

Part i. Our aim is to show that $l_{ca,t}(\theta_0)$, $l_{cb,t}(\theta_0)$, $l_{da,t}(\theta_0)$, and $l_{db,t}(\theta_0)$ all have expectation zero. Concerning $l_{ca,t}(\theta_0)$, note that $h_{a,t}(\theta_0)$, $h_{c,t}(\theta_0)$, $u_{a,t}(\theta_0)$, $h_{ca,t}(\theta_0)$, and σ_t are all $\mathcal{F}_{t-1}^{\eta}$–measurable, whereas $\mathsf{E}_{1,t}$, $\mathsf{E}_{2,t}$, and $\mathsf{E}_{3,t}$ only depend on η_t. Using parts (vii), (viii), and (ix) of Lemma D.1 we conclude that

$$E[l_{ca,t}(\theta_0)] = \frac{1}{4}\left(1 - E\left[e_{x,t}^2\eta_t^2\right]\right) E\left[\frac{h_{c,t}(\theta_0)}{\sigma_t^2}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2}\right].$$

As shown when considering the corresponding block of the covariance matrix of the score, the latter expectation equals zero.

Now consider $l_{cb,t}(\theta_0)$ and the first of its three terms. Making use of the expansion of $h_{b_q,t}(\theta_0)$ in (12d) and the fact that $h_{c,t}(\theta_0) = (1, \varepsilon_{t-1}^2, \ldots, \varepsilon_{t-R}^2)$, the components of the first term of $l_{cb,t}(\theta_0)$ can be written as ($r = 1, \ldots, R$, $q = 1, \ldots, Q$)

$$2\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\frac{\sigma_{t-\tilde{r}}\sigma_{t-\tilde{r}+q+j}}{\sigma_t^4}\eta_{t-\tilde{r}}\eta_{t-\tilde{r}+q+j}\varepsilon_{t-r}^2\mathsf{E}_{1,t}$$

or as the same expression but omitting ε_{t-r}^2 (the last option corresponding to the element 1 of $h_{c,t}(\theta_0)$). In any case, the largest time index appearing in the summands is either t or $t - \tilde{r} + q + j$. In the case $t < t - \tilde{r} + q + j$, each summand is an odd function of $\eta_{t-\tilde{r}+q+j}$, and thus the entire expression has expectation zero. If $t > t - \tilde{r} + q + j$, each summand is necessarily an odd function of either $\eta_{t-\tilde{r}}$ or $\eta_{t-\tilde{r}+q+j}$. Finally, if $t = t - \tilde{r} + q + j$, the variable $\eta_t\mathsf{E}_{1,t}$ is independent of the other variables involved and has expectation zero by Lemma D.1(x). Therefore, the first term of $l_{cb,t}(\theta_0)$ has expectation zero. Now consider the second and use the expansion of $u_{b_q,t}(\theta_0)$ in (12b) to write it as ($r = 1, \ldots, R$, $q = 1, \ldots, Q$)

$$\sum_{j=0}^{\infty}\psi_{0,j}^{(b)}\frac{\sigma_{t+q+j}}{\sigma_t^3}\eta_{t+q+j}\varepsilon_{t-r}^2\mathsf{E}_{2,t}$$

or as the same expression but omitting ε_{t-r}^2. Clearly, each summand is an odd function of η_{t+q+j}, and hence the sum has expectation zero. Concerning the third term of $l_{cb,t}(\theta_0)$, the nonzero elements of it can be written as ($r = 1, \ldots, R$, $q = 1, \ldots, Q$) $\mathsf{E}_{3,t}\sigma_t^{-2}2\varepsilon_{t-r}u'_{b_q,t-r}(\theta_0)$, or using the expansion of

$u_{b_q,t-r}(\theta_0)$ in (12b), as ($r = 1, \ldots, R$, $q = 1, \ldots, Q$)

$$2\sum_{j=0}^{\infty} \psi_{0,j}^{(b)} \frac{\sigma_{t-r+q+j}}{\sigma_t^2} \mathsf{E}_{3,t} \varepsilon_{t-r} \eta_{t-r+q+j}.$$

If $t \neq t - r + q + j$, the expression is always an odd function of $\eta_{t-r+q+j}$, and if $t = t - r + q + j$, the variable $\eta_t \mathsf{E}_{3,t}$ is independent of the other variables involved and has expectation zero by Lemma D.1(xii). Therefore, $l_{cb,t}(\theta_0)$ has expectation zero.

Now consider $l_{da,t}(\theta_0)$. Because $h_{a,t}(\theta_0)$, $u_{a,t}(\theta_0)$, and σ_t are $\mathcal{F}_{t-1}^{\eta}$–measurable,

$$E[l_{da,t}(\theta_0)] = E[e_{\lambda x,t}] E\left[\frac{u_{a,t}'(\theta_0)}{\sigma_t}\right] - \frac{1}{2} E[e_{\lambda x,t}\eta_t] E\left[\frac{h_{a,t}'(\theta_0)}{\sigma_t^2}\right].$$

As shown when considering the corresponding block of the covariance matrix of the score, both $u_{a,t}(\theta_0)/\sigma_t$ and $h_{a,t}(\theta_0)/\sigma_t^2$ have expectation zero. Hence, $E[l_{da,t}(\theta_0)] = 0$.

Finally, consider $l_{db,t}(\theta_0)$. Using the expansion of $u_{b_q,t}(\theta_0)$ in (12b), the first term contains the vectors ($q = 1, \ldots, Q$)

$$\sum_{j=0}^{\infty} \psi_{0,j}^{(b)} \frac{\sigma_{t+q+j}}{\sigma_t} e_{\lambda x,t} \eta_{t+q+j},$$

where each summand is an odd function of η_{t+q+j}. The second term contains the vectors ($q = 1, \ldots, Q$)

$$-\sum_{r=1}^{R}\sum_{j=0}^{\infty} \alpha_{0,r} \psi_{0,j}^{(b)} \frac{\sigma_{t-r}\sigma_{t-r+q+j}}{\sigma_t^2} \eta_{t-r}\eta_{t-r+q+j}\eta_t e_{\lambda x,t}.$$

If $t \neq t - r + q + j$, the expression is always an odd function of $\eta_{t-r+q+j}$, and if $t = t - r + q + j$, an odd function of η_{t-r}. Hence, $E[l_{db,t}(\theta_0)] = 0$, and Part i is completed.

Part ii. Now consider the lower right-hand corner containing the expectations of $l_{cc,t}(\theta_0)$, $l_{dc,t}(\theta_0)$, and $l_{dd,t}(\theta_0)$. Our aim is to show that these terms have expectations that equal -1 times the corresponding term in the covariance matrix of the score. Concerning the first one, note that $h_{cc,t}(\theta_0)$ is a matrix of zeros, whereas $\mathsf{E}_{1,t}$ is independent of $h_{c,t}(\theta_0)$ and σ_t^2 with expectation $E[\mathsf{E}_{1,t}] = \frac{1}{4}\left(1 - E\left[e_{x,t}^2\eta_t^2\right]\right)$ (Lemma D.1(vii)). Therefore,

$$E[l_{cc,t}(\theta_0)] = \frac{1}{4}\left(1 - E\left[e_{x,t}^2\eta_t^2\right]\right) E\left[\frac{h_{c,t}(\theta_0)}{\sigma_t^2}\frac{h_{c,t}'(\theta_0)}{\sigma_t^2}\right],$$

which indeed is the negative the matrix $\mathbf{A}_{33}$ given in Lemma 2. Concerning the second one, write $l_{dc,t}(\theta_0)$ as

$$l_{dc,t}(\theta_0) = -\frac{1}{2}\eta_t \frac{f_{\eta,\lambda x}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)} \frac{h_{c,t}'(\theta_0)}{\sigma_t^2} + \frac{1}{2} e_{\lambda,t} e_{x,t} \eta_t \frac{h_{c,t}'(\theta_0)}{\sigma_t^2}.$$

The former of these terms has expectation zero due to Lemma D.1(iv) and the fact that $h_{c,t}(\theta_0)$ and σ_t^2 are $\mathcal{F}_{t-1}^{\eta}$–measurable. The latter term has an expectation that equals -1 times the matrix $\mathbf{A}_{43}$ given in Lemma 2. Concerning the third term, as

$$l_{dd,t}(\theta_0) = e_{\lambda\lambda,t} = \frac{f_{\eta,\lambda\lambda}(\eta_t;\lambda)}{f_\eta(\eta_t;\lambda)} - e_{\lambda,t}e_{\lambda,t}',$$

Lemma D.1(ii) immediately shows that $E[l_{dd,t}(\theta_0)]$ equals the negative of the matrix $\mathbf{A}_{44}$ given in Lemma 2. This completes Part ii.

Part iii. Now consider the top left-hand corner containing the expectations of $l_{aa,t}(\theta_0)$, $l_{ba,t}(\theta_0)$, and $l_{bb,t}(\theta_0)$. Again, our aim is to show that these terms have expectations that equal -1 times the corresponding term in the covariance matrix of the score. Concerning $l_{aa,t}(\theta_0)$, note that each of its six terms is a product of a part that only depends on η_t, and of a part that is independent of η_t. Using parts (v), (vii), (viii), and (ix) of Lemma D.1 and the fact that $E[e_{x,t}] = 0$, it follows that

$$E[l_{aa,t}(\theta_0)] = -E[e_{x,t}^2]E\left[\frac{u_{a,t}(\theta_0)}{\sigma_t}\frac{u'_{a,t}(\theta_0)}{\sigma_t}\right] + \frac{1}{4}\left(1 - E\left[e_{x,t}^2\eta_t^2\right]\right)E\left[\frac{h_{a,t}(\theta_0)}{\sigma_t^2}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2}\right],$$

which indeed equals -1 times the matrix $\mathbf{A}_{11}$ given in Lemma 2.

Now consider the expectation of $l_{ba,t}(\theta_0)$. Recall that

$$\begin{aligned} l_{ba,t}(\theta_0) \;=\;& e_{xx,t}\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u'_{a,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{1,t}\frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2} \\ &+\mathsf{E}_{2,t}\left(\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{h'_{a,t}(\theta_0)}{\sigma_t^2} + \frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{u'_{a,t}(\theta_0)}{\sigma_t}\right) + e_{x,t}\frac{u_{ba,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{3,t}\frac{h_{ba,t}(\theta_0)}{\sigma_t^2}. \end{aligned}$$

We will show that out of the six terms, (i) the first and the third have expectation zero, (ii) the second and the fourth will yield the term $-\mathbf{A}_{21}$, (iii) the fifth term yields the term $-\mathbf{B}_{21}$, and finally that (iv) also the sixth term has expectation zero.

(i) Using the expansions in (12) we can express typical components of the first and third terms as

$$\begin{aligned} &-\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi_{0,i}^{(a)}\psi_{0,j}^{(b)}\frac{\sigma_{t-p-i}\sigma_{t+q+j}}{\sigma_t^2}\eta_{t-p-i}\eta_{t+q+j}e_{xx,t} \\ &-2\sum_{r=1}^{R}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{0,r}\psi_{0,i}^{(a)}\psi_{0,j}^{(b)}\frac{\sigma_{t-r}\sigma_{t-r-p-i}\sigma_{t+q+j}}{\sigma_t^3}\eta_{t-r}\eta_{t-r-p-i}\eta_{t+q+j}\mathsf{E}_{2,t}. \end{aligned}$$

It is clear that in both expressions the summands are odd functions of η_{t+q+j}, and hence both of them have expectation zero.

(ii) First note that the matrix $\mathbf{A}_{21}$ contains expectations of terms that are exactly the same as the second and the fourth term except for the fact that instead of the multipliers $\mathsf{E}_{1,t}$ and $\mathsf{E}_{2,t}$, they contain the multipliers $\frac{1}{4}(e_{x,t}\eta_t+1)^2$ and $-\frac{1}{2}e_{x,t}(e_{x,t}\eta_t+1)$. As can be seen from Lemma D.1(vii) and (xi), it holds that $E[\mathsf{E}_{1,t}] = -E[\frac{1}{4}(e_{x,t}\eta_t+1)^2]$ and $E[\eta_t\mathsf{E}_{2,t}] = -E[-\frac{1}{2}e_{x,t}\eta_t(e_{x,t}\eta_t+1)]$. Keeping these facts in mind, we can now express the typical elements of the second and the fourth term as

$$\begin{aligned} &-4\sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,i}^{(a)}\psi_{0,j}^{(b)}\frac{\sigma_{t-r}\sigma_{t-\tilde{r}}\sigma_{t-r+q+j}\sigma_{t-\tilde{r}-p-i}}{\sigma_t^4}\eta_{t-\tilde{r}}\eta_{t-\tilde{r}-p-i}\eta_{t-r}\eta_{t-r+q+j}\mathsf{E}_{1,t} \\ &-2\sum_{r=1}^{R}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\alpha_{0,r}\psi_{0,i}^{(a)}\psi_{0,j}^{(b)}\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{t-p-i}}{\sigma_t^3}\eta_{t-r}\eta_{t-r+q+j}\eta_{t-p-i}\mathsf{E}_{2,t}. \end{aligned}$$

Concerning the former expression, the largest time index in the summands is either t or $t-r+q+j$. If $t<t-r+q+j$ or $t=t-r+q+j$, either $\eta_{t-r+q+j}$ or $\eta_t \mathsf{E}_{1,t}$ is independent of the other variables involved and has zero expectation (by Lemma D.1(x) in the latter case). If $t>t-r+q+j$, the variable $\mathsf{E}_{1,t}$ is independent of the other variables involved and, therefore, its expectation can be computed separately. In light of the these facts, we can thus conclude that this expression indeed has an expectation that equals -1 times the first term of $\mathbf{A}_{21}$. Now, consider the latter expression and note that the largest time index in the summands is either t or $t-r+q+j$. If $t \neq t-r+q+j$, one of the variables $\eta_{t-r+q+j}$ and $\mathsf{E}_{2,t}$ is independent of the other variables involved and has zero expectation (by Lemma D.1(viii) in the latter case). If $t=t-r+q+j$, the variable $\eta_t \mathsf{E}_{2,t}$ is independent of the other variables involved and, therefore, its expectation can be computed separately. As above, we obtain -1 times the second term of $\mathbf{A}_{21}$. This completes the proof of (ii).

(iii) We next show that the fifth term, namely $e_{x,t}\frac{u_{ba,t}(\theta_0)}{\sigma_t}$, yields the term $-\mathbf{B}_{21}$. To this end, we first derive an alternative expression for $(\mathbf{B}_{21})_{q,p}$. Note that assuming $q<p$, solving for j, and substituting shows that $(\mathbf{B}_{21})_{q,p} = -\sum_{i=0}^{\infty} \psi_{0,i+p-q}^{(b)} \psi_{0,i}^{(a)}$. Now define $\psi_{0,j}^{(ab)}$, $j \in \mathbb{Z}$, as the coefficient of z^j in the power series expansion of $a_0(z)^{-1} b_0(z^{-1})^{-1}$, that is,

$$\begin{aligned} a_0(z)^{-1} b_0(z^{-1})^{-1} &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \psi_{0,j}^{(a)} \psi_{0,k}^{(b)} z^j z^{-k} \\ &= \sum_{l=0}^{\infty} \psi_{0,l}^{(a)} \psi_{0,l}^{(b)} z^0 + \sum_{l=0}^{\infty} \psi_{0,l+1}^{(a)} \psi_{0,l}^{(b)} z^1 + \sum_{l=0}^{\infty} \psi_{0,l}^{(a)} \psi_{0,l+1}^{(b)} z^{-1} + \dots \\ &= \psi_{0,0}^{(ab)} z^0 + \psi_{0,1}^{(ab)} z^1 + \psi_{0,-1}^{(ab)} z^{-1} + \dots \end{aligned}$$

It is seen that with this notation we can write $(\mathbf{B}_{21})_{q,p} = -\psi_{0,q-p}^{(ab)}$.

Now recall that the typical element of $u_{ba,t}(\theta)$ is given by ($q=1,\dots,Q$, $p=1,\dots,P$)

$$u_{b_q a_p,t}(\theta) = \frac{\partial^2 u_t(\theta)}{\partial b_q \partial a_p} = -b\left(B^{-1}\right)^{-1} a(B)^{-1} u_{t+q-p}(\theta)$$

so that the typical element of $u_{ba,t}(\theta_0)$ is $-b_0\left(B^{-1}\right)^{-1} a_0(B)^{-1}\varepsilon_{t+q-p}$. Therefore, we can write the typical element of $u_{ba,t}(\theta_0)$ as

$$-\sum_{j=-\infty}^{\infty} \psi_{0,j}^{(ab)} \varepsilon_{t+q-p-j} = -\sum_{j=-\infty}^{\infty} \psi_{0,j}^{(ab)} \sigma_{t+q-p-j}\eta_{t+q-p-j}.$$

Thus, the typical element of $e_{x,t}\frac{u_{ba,t}(\theta_0)}{\sigma_t}$ can be expressed as ($q=1,\dots,Q$, $p=1,\dots,P$)

$$-\sum_{j=-\infty}^{\infty} \psi_{0,j}^{(ab)} \frac{\sigma_{t+q-p-j}}{\sigma_t} \eta_{t+q-p-j} e_{x,t}.$$

Note that all summands have zero expectation except for the one with $j=q-p$, and therefore

$$E\left[e_{x,t}\frac{u_{b_q a_p,t}(\theta_0)}{\sigma_t}\right] = \psi_{0,q-p}^{(ab)}, \quad q=1,\dots,Q, \ p=1,\dots,P,$$

which indeed are -1 times the components of $\mathbf{B}_{21}$.

(iv) Finally, we establish that the sixth term, $\mathsf{E}_{3,t}\frac{h_{ba,t}(\theta_0)}{\sigma_t^2}$, has expected value zero. Making use of the expansion of $h_{ba,t}(\theta)$ we obtain

$$\mathsf{E}_{3,t}\frac{h_{ba,t}(\theta_0)}{\sigma_t^2} = 2\sum_{r=1}^{R}\alpha_{0,r}\frac{u_{b,t-r}(\theta_0)}{\sigma_t}\frac{u'_{a,t-r}(\theta_0)}{\sigma_t}\mathsf{E}_{3,t} + 2\sum_{r=1}^{R}\alpha_{0,r}\varepsilon_{t-r}\frac{u_{ba,t-r}(\theta_0)}{\sigma_t^2}\mathsf{E}_{3,t}.$$

Consider the first sum. For $r = 1,\ldots,R$, the typical element of the summand can be written as

$$-\alpha_{0,r}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\psi_{0,i}^{(a)}\psi_{0,j}^{(b)}\frac{\sigma_{t-r+q+j}\sigma_{t-r-p-i}}{\sigma_t^2}\eta_{t-r+q+j}\eta_{t-r-p-i}\mathsf{E}_{3,t},$$

where the summands are always odd functions of $\eta_{t-r-p-i}$. Hence the sum has expectation zero. Now consider the second sum. For $r = 1,\ldots,R$, the typical element of the summand can be written as

$$-\alpha_{0,r}\sum_{j=-\infty}^{\infty}\psi_{0,j}^{(ab)}\frac{\sigma_{t-r+q-p-j}\sigma_{t-r}}{\sigma_t^2}\eta_{t-r+q-p-j}\eta_{t-r}\mathsf{E}_{3,t},$$

which is an odd function of η_{t-r} unless $j = q-p$. However, if $j = q-p$ the variable $\mathsf{E}_{3,t}$ is independent of the other variables involved and has expectation zero (by Lemma D.1(ix)). Thus, we have established that the sixth term has expectation zero. This completes the proof of (iv), and the computations concerning $l_{ba,t}(\theta_0)$.

Finally, consider the expectation of $l_{bb,t}(\theta_0)$. Recall that

$$\begin{aligned} l_{bb,t}(\theta_0) &= e_{xx,t}\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u'_{b,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{1,t}\frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{h'_{b,t}(\theta_0)}{\sigma_t^2} \\ &\quad + \mathsf{E}_{2,t}\left(\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{h'_{b,t}(\theta_0)}{\sigma_t^2} + \frac{h_{b,t}(\theta_0)}{\sigma_t^2}\frac{u'_{b,t}(\theta_0)}{\sigma_t}\right) + e_{x,t}\frac{u_{bb,t}(\theta_0)}{\sigma_t} + \mathsf{E}_{3,t}\frac{h_{bb,t}(\theta_0)}{\sigma_t^2}. \end{aligned}$$

We first consider the third, fourth, and fifth terms and show that they have expected value zero. The typical element of the third term can be expressed as ($q = 1,\ldots,Q$, $\tilde{q} = 1,\ldots,Q$)

$$2\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t+q+j}\sigma_{t-r}\sigma_{t-r+\tilde{q}+\tilde{j}}}{\sigma_t^3}\eta_{t+q+j}\eta_{t-r}\eta_{t-r+\tilde{q}+\tilde{j}}\mathsf{E}_{2,t},$$

where each summand is always an odd function of η_{t-r}. Hence, the third term has expectation zero. With nearly identical reasoning, also the fourth term has expectation zero. Now consider the fifth term, and recall that the typical element of $u_{bb,t}(\theta)$ is given by ($q = 1,\ldots,Q$, $\tilde{q} = 1,\ldots,Q$)

$$u_{b_q b_{\tilde{q}},t}(\theta) = \frac{\partial^2 u_t(\theta)}{\partial b_q \partial b_{\tilde{q}}} = 2b\left(B^{-1}\right)^{-2}u_{t+q+\tilde{q}}(\theta),$$

so that the typical element of $u_{bb,t}(\theta_0)$ is $2b_0\left(B^{-1}\right)^{-2}\varepsilon_{t+q+\tilde{q}}$. This can be written as

$$2\sum_{j=0}^{\infty}\psi_{0,j}^{(bb)}\sigma_{t+q+\tilde{q}+j}\eta_{t+q+\tilde{q}+j},$$

where $\psi_{0,j}^{(bb)}$ denotes the coefficient of z^j in the series expansion of $b_0\left(z^{-1}\right)^{-2}$. Hence, the typical element of the fifth term can be written as

$$2\sum_{j=0}^{\infty}\psi_{0,j}^{(bb)}\frac{\sigma_{t+q+\tilde{q}+j}}{\sigma_t}\eta_{t+q+\tilde{q}+j}e_{x,t},$$

where each summand is an odd function of $\eta_{t+q+\tilde{q}+j}$. Thus, the fifth term has expectation zero.

What remains to consider are the first, second, and sixth terms. We begin with the second term, and show that it yields -1 times the latter term of $\mathbf{A}_{22}$. The typical element of the second term can be written as

$$4\sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\alpha_{0,\tilde{r}}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t-r}\sigma_{t-r+q+j}\sigma_{t-\tilde{r}}\sigma_{t-\tilde{r}+\tilde{q}+\tilde{j}}}{\sigma_t^4}\eta_{t-r}\eta_{t-r+q+j}\eta_{t-\tilde{r}}\eta_{t-\tilde{r}+\tilde{q}+\tilde{j}}\mathsf{E}_{1,t}.$$

To avoid an odd expression as a function of one of the η's, we must have $r=\tilde{r}$ and $q+j=\tilde{q}+\tilde{j}$. Note from the definitions of $\mathsf{E}_{1,t}$, $e_{xx,t}$, and $\mathsf{E}_{4,t}$ that $\mathsf{E}_{1,t}=\mathsf{E}_{4,t}-\frac{1}{4}\left(e_{x,t}\eta_t+1\right)^2$. From the preceding discussion it follows that the considered term yields -1 times the latter term in $\mathbf{A}_{22}$ if (recall the restrictions $r=\tilde{r}$ and $q+j=\tilde{q}+\tilde{j}$)

$$4\sum_{r=1}^{R}\sum_{\tilde{r}=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}^2\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t-r}^2\sigma_{t-r+q+j}^2}{\sigma_t^4}\eta_{t-r}^2\eta_{t-r+q+j}^2\mathsf{E}_{4,t}$$

has expectation zero. By Lemma D.1(xiv) and (xvi), $E\left[\mathsf{E}_{4,t}\right]=E\left[\eta_t^2\mathsf{E}_{4,t}\right]=0$. If $t-r+q+j<t$, the variable $\mathsf{E}_{4,t}$ is independent of the other variables involved and has zero expectation. Thus, the sum has zero expected value and we can assume $t-r+q+j\overset{def}{=}t+a\geq t$. As $E\left[\mathsf{E}_{4,t}\right]=E\left[\eta_t^2\mathsf{E}_{4,t}\right]=0$, zero expectation is obtained by Lemma B.1, which completes the treatment of the second term.

Now consider the first term,

$$e_{xx,t}\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u_{b,t}'(\theta_0)}{\sigma_t}=\frac{f_{\eta,xx}\left(\eta_t;\lambda_0\right)}{f_\eta\left(\eta_t;\lambda_0\right)}\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u_{b,t}'(\theta_0)}{\sigma_t}-e_{x,t}^2\frac{u_{b,t}(\theta_0)}{\sigma_t}\frac{u_{b,t}'(\theta_0)}{\sigma_t}.$$

The latter term obviously produces -1 times the first term of $\mathbf{A}_{22}$. The typical element of the former term can be written as

$$\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t+q+j}\sigma_{t+\tilde{q}+\tilde{j}}}{\sigma_t^2}\eta_{t+q+j}\eta_{t+\tilde{q}+\tilde{j}}\frac{f_{\eta,xx}\left(\eta_t;\lambda_0\right)}{f_\eta\left(\eta_t;\lambda_0\right)}.$$

To avoid an odd expression as a function of η_{t+q+j} or $\eta_{t+\tilde{q}+\tilde{j}}$, we must have $q+j=\tilde{q}+\tilde{j}\overset{def}{=}a\ (\geq 1)$. By Assumption 5, $E\left[f_{\eta,xx}\left(\eta_t;\lambda_0\right)/f_\eta\left(\eta_t;\lambda_0\right)\right]=0$ and $E\left[\eta_t^2f_{\eta,xx}\left(\eta_t;\lambda_0\right)/f_\eta\left(\eta_t;\lambda_0\right)\right]=2$, and therefore by Lemma B.1 this expression has expectation equal to

$$2\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}c_a\mathbf{1}\left(q+j=\tilde{q}+\tilde{j}=a\geq 1\right).$$

This is $-\frac{1}{2}$ times the expression given in Lemma 2 for the components of $\mathbf{B}_{22}$.

To complete the proof, we must show that also the sixth term yields $-\frac{1}{2}$ times the expression given in Lemma 2 for the components of $\mathbf{B}_{22}$. Using the expansion of $h_{bb,t}(\theta)$ we obtain

$$\mathsf{E}_{3,t}\frac{h_{bb,t}(\theta_0)}{\sigma_t^2} = 2\sum_{r=1}^{R}\alpha_{0,r}\frac{u_{b,t-r}(\theta_0)}{\sigma_t}\frac{u'_{b,t-r}(\theta_0)}{\sigma_t}\mathsf{E}_{3,t} + 2\sum_{r=1}^{R}\alpha_{0,r}\varepsilon_{t-r}\frac{u_{bb,t-r}(\theta_0)}{\sigma_t^2}\mathsf{E}_{3,t}.$$

First consider the latter sum. For $r = 1, \ldots, R$, the typical element of the summand can be written as ($q = 1, \ldots, Q$, $\tilde{q} = 1, \ldots, Q$)

$$2\alpha_{0,r}\sum_{j=0}^{\infty}\psi_{0,j}^{(bb)}\frac{\sigma_{t-r+q+\tilde{q}+j}\sigma_{t-r}}{\sigma_t^2}\eta_{t-r+q+\tilde{q}+j}\eta_{t-r}\mathsf{E}_{3,t}.$$

As each summand is clearly an odd function of η_{t-r}, the latter sum has expectation zero. Now consider the first sum, whose typical element can be written as ($q = 1, \ldots, Q$, $\tilde{q} = 1, \ldots, Q$)

$$2\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t-r+q+j}\sigma_{t-r+\tilde{q}+\tilde{j}}}{\sigma_t^2}\eta_{t-r+q+j}\eta_{t-r+\tilde{q}+\tilde{j}}\mathsf{E}_{3,t}.$$

To avoid an odd expression as a function of $\eta_{t-r+q+j}$ or $\eta_{t-r+\tilde{q}+\tilde{j}}$, we must have $t-r+q+j = t-r+\tilde{q}+\tilde{j}$. When $t > t-r+q+j$ the variable $\mathsf{E}_{3,t}$ is independent of $\sigma_{t-r+q+j}^2$ and $\eta_{t-r+q+j}^2$ (and σ_t^2) and has expectation zero (by Lemma D.1(ix)). Therefore, we can assume the restriction $t \leq t-r+q+j \overset{def}{=} t+a \geq t$. Then the above expression can be expressed as

$$2\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}\frac{\sigma_{t+a}^2}{\sigma_t^2}\eta_{t+a}^2\mathsf{E}_{3,t},$$

where the restrictions $q+j = \tilde{q}+\tilde{j} = r+a$ are assumed. By Lemma D.1(ix) and (xiii), $E\left[\mathsf{E}_{3,t}\right] = 0$ and $E\left[\eta_t^2\mathsf{E}_{3,t}\right] = 1$, so that using Lemma B.1, the expectation of the above expression equals

$$2\sum_{r=1}^{R}\sum_{j=0}^{\infty}\sum_{\tilde{j}=0}^{\infty}\alpha_{0,r}\psi_{0,j}^{(b)}\psi_{0,\tilde{j}}^{(b)}c_a\mathbf{1}\left(q+j-r = \tilde{q}+\tilde{j}-r = a\right)$$

(recall that $c_a = 0$ for $a < 0$ so all values of a can be included above). This is indeed $-\frac{1}{2}$ times the expression given in Lemma 2 for the components of $\mathbf{B}_{22}$. This completes the computations concerning the term $l_{bb,t}(\theta_0)$, and hence the proof is complete. ∎

Proof of Lemma D.1. (i) Already proven as Lemma C.1(xi). (ii) Note that

$$E\left[\frac{f_{\eta,\lambda\lambda}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\right] = \int f_{\eta,\lambda\lambda}(x;\lambda_0)dx = \int\left[\frac{\partial^2 f_\eta(x;\lambda)}{\partial\lambda\partial\lambda'}\right]_{\lambda=\lambda_0}dx = \left[\frac{d^2}{d\lambda d\lambda'}\int f_\eta(x;\lambda)dx\right]_{\lambda=\lambda_0} = 0,$$

where the second last equality is justified by Assumption 5(i), cf. proof of Lemma C.1(xi). (iii) This holds because

$$E\left[\eta_t e_{\lambda,t}\right] = \int x f_{\eta,\lambda}(x;\lambda_0)dx = \int\left[\frac{\partial\left(x f_\eta(x;\lambda)\right)}{\partial\lambda}\right]_{\lambda=\lambda_0}dx = \left[\frac{d}{d\lambda}\int x f_\eta(x;\lambda)dx\right]_{\lambda=\lambda_0} = 0,$$

where the second last equality is justified by Assumption 4(v), cf. proof of Lemma C.1(xi), and the last equality by Assumption 4(ii). (iv) Use integration by parts to obtain

$$E\left[\eta_t \frac{f_{\eta,\lambda x}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\right] = \int x f_{\eta,\lambda x}(x;\lambda_0)\,dx = |_{-\infty}^{\infty} x f_{\eta,\lambda}(x;\lambda_0) - \int f_{\eta,\lambda}(x;\lambda_0)\,dx.$$

The integral on the right is zero as it equals $E[e_{\lambda,t}]$, whereas the first term on the right is zero because $E[\eta_t e_{\lambda,t}] = 0$. (v) Recall that $e_{xx,t} = f_{\eta,xx}(\eta_t;\lambda_0)/f_\eta(\eta_t;\lambda_0) - e_{x,t}^2$. By Assumption 5(ii), $\int f_{\eta,xx}(x;\lambda_0)\,dx = 0$ so that $E[e_{xx,t}] = -E[e_{x,t}^2]$. (vi) Note that $e_{xx,t}\eta_t = \frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t - e_{x,t}^2\eta_t$. The latter term is an odd function of η_t and thus has expectation zero by Lemma A.3 and the fact that $E[e_{x,t}^2|\eta_t|] < \infty$ (which follows from Lemma C.1). The former term is also an odd function of η_t (the reason for this being that the second derivative of a function symmetric around zero is also symmetric around zero), and thus also has expectation zero by Lemma A.3 (that $E\left[\left|\frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t\right|\right]$ is finite follows from part (xv) of this lemma). (vii) Write $\mathsf{E}_{1,t}$ as

$$\mathsf{E}_{1,t} = \frac{1}{4}\frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t^2 - \frac{1}{4}e_{x,t}^2\eta_t^2 + \frac{3}{4}e_{x,t}\eta_t + \frac{1}{2},$$

where $E\left[\frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t^2\right] = 2$ by Assumption 5(iii) and $E[e_{x,t}\eta_t] = -1$ (see Lemma C.1(viii)). Therefore $E[\mathsf{E}_{1,t}] = \frac{1}{4}(1 - E[e_{x,t}^2\eta_t^2])$. (viii) Because $e_{xx,t}\eta_t$ and $e_{x,t}$ both have zero expectation (former by part (vi)), $E[\mathsf{E}_{2,t}] = 0$. (ix) Clear because $E[e_{x,t}\eta_t] = -1$. (x) Write $\eta_t\mathsf{E}_{1,t}$ as

$$\eta_t\mathsf{E}_{1,t} = \frac{1}{2}\left(\frac{1}{2}e_{xx,t}\eta_t^3 + \frac{3}{2}e_{x,t}\eta_t^2 + \eta_t\right) = \frac{1}{2}\left(\frac{1}{2}\frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t^3 - \frac{1}{2}e_{x,t}^2\eta_t^3 + \frac{3}{2}e_{x,t}\eta_t^2 + \eta_t\right).$$

As in part (vi), the first term is an odd function of η_t, and so are the three last terms. Hence, $E[\eta_t\mathsf{E}_{1,t}] = 0$ by Lemma A.3 (that the terms are integrable follows by part xv of this lemma and Lemma C.1). (xi) Write $\eta_t\mathsf{E}_{2,t}$ as

$$\eta_t\mathsf{E}_{2,t} = -\frac{1}{2}\left(\frac{f_{\eta,xx}(\eta_t;\lambda)}{f_\eta(\eta_t;\lambda)}\eta_t^2 - e_{x,t}^2\eta_t^2 + e_{x,t}\eta_t\right).$$

As noted in part (vii), $E\left[\frac{f_{\eta,xx}(\eta_t;\lambda)}{f_\eta(\eta_t;\lambda)}\eta_t^2\right] = 2$, so that $E[\eta_t\mathsf{E}_{2,t}] = -\frac{1}{2}(1 - E[e_{x,t}^2\eta_t^2])$. (xii) Clear because both $e_{x,t}\eta_t^2$ and η_t have expectation zero (former by Lemma C.1(ix)). (xiii) By the definition of $\mathsf{E}_{3,t}$

$$E\left[\eta_t^2\mathsf{E}_{3,t}\right] = -\frac{1}{2}E\left[\eta_t^2(e_{x,t}\eta_t + 1)\right] = -\frac{1}{2}E\left[\eta_t^3 e_{x,t}\right] - \frac{1}{2}E\left[\eta_t^2\right].$$

The stated result follows because $E\left[\eta_t^2\right] = 1$ and, by Lemma C.1(x), $E\left[e_{x,t}\eta_t^3\right] = -3$. (xiv) Follows by Assumption 5(iii) and the fact that $E[e_{x,t}\eta_t] = -1$. (xv) Using integration by parts twice

$$\begin{aligned} E\left[\frac{f_{\eta,xx}(\eta_t;\lambda_0)}{f_\eta(\eta_t;\lambda_0)}\eta_t^4\right] &= \int x^4 f_{\eta,xx}(x;\lambda_0)\,dx = |_{-\infty}^{\infty} x^4 f_{\eta,x}(x;\lambda_0) - 4\int x^3 f_{\eta,x}(x;\lambda_0)\,dx \\ &= -4|_{-\infty}^{\infty} x^3 f_\eta(x;\lambda_0) + 12\int x^2 f_\eta(x;\lambda_0)\,dx = 12 \end{aligned}$$

where the third equality also relies on the fact that $E\left[\eta_t^4 e_{x,t}\right]=0$ (which holds by Lemma A.3 because the argument is an odd function of η_t and because $E\left[\eta_t^4\left|e_{x,t}\right|\right]$ is finite as a consequence of Lemma C.1(i)). The fourth equality is due to the facts that $E\left[\eta_t^3\right]=0$ and $E\left[\eta_t^2\right]=1$. (xvi) Follows using the previous part and the result $E\left[e_{x,t}\eta_t^3\right]=-3$ (see Lemma C.1(x)). ■

Proof of Lemma E.1. (i) From the representations of $\tilde{u}_t(\theta)$ and $u_t(\theta)$ in (24) and (25) we obtain $u_t(\theta)-\tilde{u}_t(\theta)=\sum_{j=T+1-t}^{\infty}\psi_j^{(b)}a(B)y_{t+j}$ so that

$$\left\|\sup_{\theta\in\Theta_0}\left|u_t(\theta)-\tilde{u}_t(\theta)\right|\right\|_4\leq\sum_{j=T+1-t}^{\infty}\sup_{\theta\in\Theta_0}\left|\psi_j^{(b)}\right|\left\|\sup_{\theta\in\Theta_0}\left|a(B)y_{t+j}\right|\right\|_4.$$

By Lemma A.1 and the discussion following it, $\sup_{\theta\in\Theta_0}\left|\psi_j^{(b)}\right|\leq C_1\rho^j$. On the other hand, because $E\left[y_t^4\right]<\infty$ (by Lemma 2) and Θ_0 is compact, it clearly holds that $\left\|\sup_{\theta\in\Theta_0}\left|a(B)y_{t+j}\right|\right\|_4\leq C_2$. Therefore, the sum on the majorant side is bounded from above by

$$\sum_{j=T+1-t}^{\infty}C_1C_2\rho^j=\rho^{T+1-t}\frac{C_1C_2}{1-\rho},$$

from which the result follows with a suitable choice of C.

(ii) Using the inequality $\left|u_t^2(\theta)-\tilde{u}_t^2(\theta)\right|\leq\left|u_t(\theta)-\tilde{u}_t(\theta)\right|^2+2\left|u_t(\theta)\right|\left|u_t(\theta)-\tilde{u}_t(\theta)\right|$ we have

$$\left\|\sup_{\theta\in\Theta_0}\left|u_t^2(\theta)-\tilde{u}_t^2(\theta)\right|\right\|_2\leq\left\|\sup_{\theta\in\Theta_0}\left|u_t(\theta)-\tilde{u}_t(\theta)\right|\right\|_4^2+2\left\|\sup_{\theta\in\Theta_0}\left|u_t(\theta)\right|\right\|_4\left\|\sup_{\theta\in\Theta_0}\left|u_t(\theta)-\tilde{u}_t(\theta)\right|\right\|_4.$$

To obtain the result, use part (i) on the majorant side in conjunction with the fact $\rho^2<\rho$ and the fact $\left\|\sup_{\theta\in\Theta_0}\left|u_t(\theta)\right|\right\|_4<\infty$ (established in the proof of Lemma 6),.

(iii) From the definitions of $h_t(\theta)$ and $\tilde{h}_t(\theta)$ in (9) and (10) it follows that $\left|h_t(\theta)-\tilde{h}_t(\theta)\right|\leq\sum_{r=1}^{R}\alpha_r\left|u_{t-r}^2(\theta)-\tilde{u}_{t-r}^2(\theta)\right|$. By the definition of Θ_c in Assumption 2, $\alpha_r<1$ for $r=1,\ldots,R$, and therefore

$$\left\|\sup_{\theta\in\Theta_0}\left|h_t(\theta)-\tilde{h}_t(\theta)\right|\right\|_2\leq\sum_{r=1}^{R}\left\|\sup_{\theta\in\Theta_0}\left|u_{t-r}^2(\theta)-\tilde{u}_{t-r}^2(\theta)\right|\right\|_2.$$

For $t=R+1,\ldots,T$, the desired result follows from part (ii) by choosing a suitable C. For $t=1$,

$$\left\|\sup_{\theta\in\Theta_0}\left|h_1(\theta)-\tilde{h}_1(\theta)\right|\right\|_2\leq\sum_{r=1}^{R}\left(\left\|\sup_{\theta\in\Theta_0}\left|u_{1-r}(\theta)\right|\right\|_4^2+u_{1-r}^2\right)$$

where the majorant is finite because $\left\|\sup_{\theta\in\Theta_0}\left|u_t(\theta)\right|\right\|_4$ and the constants $u_0,\ldots,u_{1-R}$ are finite. In the remaining cases ($t=2,\ldots,R$), each of the R terms in the resulting majorant can be either (when $t-r\leq0$) bounded by a constant as in the case $t=1$, or (when $t-r\geq1$) by a term $C\rho^{T+1-t}<C$ as in the case $t=R+1,\ldots,T$. Hence, we obtain the upper bound $CU_{2,t}$.

(iv) Using the mean value theorem and the definition of Θ_c in Assumption 2 (ensuring $h_t(\theta),\tilde{h}_t(\theta)\geq\underline{\omega}>0$) we obtain

$$\sup_{\theta\in\Theta_0}\left|\log h_t(\theta)-\log\tilde{h}_t(\theta)\right|\leq C\sup_{\theta\in\Theta_0}\left|h_t(\theta)-\tilde{h}_t(\theta)\right|$$

for some finite C. Thus, the result follows from part (iii).

(v) First consider the difference $\left|h_t^{-1/2}(\theta)-\tilde{h}_t^{-1/2}(\theta)\right|$, and use the mean value theorem for the function $x^{-1/2}$ (and the above-mentioned fact that $h_t(\theta), \tilde{h}_t(\theta) \geq \underline{\omega} > 0$) to obtain $\left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right| \leq \frac{1}{2}\underline{\omega}^{-3/2}\left|h_t(\theta) - \tilde{h}_t(\theta)\right|$. This and part (iii) yield the intermediate result

$$\left\|\sup_{\theta\in\Theta_0}\left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right|\right\|_2 \leq CU_{2,t}. \tag{32}$$

Now, making use of the inequality $|xy - \tilde{x}\tilde{y}| \leq |x - \tilde{x}|\,|y| + |x - \tilde{x}|\,|y - \tilde{y}| + |x|\,|y - \tilde{y}|$, we get

$$\begin{aligned}\left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| &\leq |u_t(\theta) - \tilde{u}_t(\theta)|\left|h_t^{-1/2}(\theta)\right| + |u_t(\theta) - \tilde{u}_t(\theta)|\left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right| \\ &\quad + |u_t(\theta)|\left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right|.\end{aligned}$$

The result is obtained from this by using Hölder's inequality, the norm inequality, the fact $h_t(\theta) \geq \underline{\omega} > 0$, the result $\left\|\sup_{\theta\in\Theta_0}|u_t(\theta)|\right\|_4 < \infty$, part (i), and (32).

(vi) We start by considering the difference $|u_{a,t}(\theta) - \tilde{u}_{a,t}(\theta)|$. From the expressions of $\tilde{u}_{a_p,t}(\theta)$ and $u_{a_p,t}(\theta)$ in (26) and (28) we obtain $\left|u_{a_p,t}(\theta) - \tilde{u}_{a_p,t}(\theta)\right| \leq \sum_{j=T+1-t}^{\infty}\psi_j^{(b)}y_{t-p+j}$ so that, because $\sup_{\theta\in\Theta_0}\left|\psi_j^{(b)}\right| \leq C_1\rho^j$ and $E\left[y_t^4\right] < \infty$,

$$\left\|\sup_{\theta\in\Theta_0}|u_{a,t}(\theta) - \tilde{u}_{a,t}(\theta)|\right\|_4 \leq CU_{1,t} \tag{33}$$

for some finite C. Now, as in the proof of part (v),

$$\begin{aligned}\left|\frac{u_{a,t}(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_{a,t}(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| &\leq |u_{a,t}(\theta) - \tilde{u}_{a,t}(\theta)|\left|h_t^{-1/2}(\theta)\right| + |u_{a,t}(\theta) - \tilde{u}_{a,t}(\theta)|\left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right| \\ &\quad + |u_{a,t}(\theta)|\left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right|,\end{aligned}$$

and the result is obtained exactly as therein, except that now (33) and the result $\left\|\sup_{\theta\in\Theta_0}|u_{a,t}(\theta)|\right\|_4 < \infty$ obtained in the proof of Lemma 6 are used.

(vii) First consider the difference $\left|h_{a,t}(\theta) - \tilde{h}_{a,t}(\theta)\right|$. Making use of the fact that $\alpha_r < 1$ for $r = 1, \ldots, R$ and the inequality $|xy - \tilde{x}\tilde{y}| \leq |x - \tilde{x}|\,|y| + |x - \tilde{x}|\,|y - \tilde{y}| + |x|\,|y - \tilde{y}|$,

$$\begin{aligned}\left|h_{a,t}(\theta) - \tilde{h}_{a,t}(\theta)\right| &\leq 2\sum_{r=1}^{R}\alpha_r\left|u_{t-r}(\theta)u_{a,t-r}(\theta) - \tilde{u}_{t-r}(\theta)\tilde{u}_{a,t-r}(\theta)\right| \\ &\leq 2\sum_{r=1}^{R}|u_{t-r}(\theta) - \tilde{u}_{t-r}(\theta)|\,|u_{a,t-r}(\theta)| + 2\sum_{r=1}^{R}|u_{t-r}(\theta)|\,|u_{a,t-r}(\theta) - \tilde{u}_{a,t-r}(\theta)| \\ &\quad + 2\sum_{r=1}^{R}|u_{t-r}(\theta) - \tilde{u}_{t-r}(\theta)|\,|u_{a,t-r}(\theta) - \tilde{u}_{a,t-r}(\theta)|\,.\end{aligned}$$

For $t = 1, \ldots, R$, $\left\|\sup_{\theta\in\Theta_0}\left|h_{a,t}(\theta) - \tilde{h}_{a,t}(\theta)\right|\right\|_2$ is bounded by a finite constant (the reasoning for this is similar to that used in part (iii); now using also (33), the fact that the derivatives $\tilde{u}_{a_p,t-r}(\theta)$

equal zero when $t-r \le 0$, and the result $\left\|\sup_{\theta\in\Theta_0}|u_{a,t}(\theta)|\right\|_4 < \infty$). For $t = R+1,\ldots,T$, the upper bound $C_1\rho^{T+1-t}$ is obtained from part (i), (33), and the results $\left\|\sup_{\theta\in\Theta_0}|u_t(\theta)|\right\|_4 < \infty$ and $\left\|\sup_{\theta\in\Theta_0}|u_{a,t}(\theta)|\right\|_4 < \infty$. Thus, we obtain the intermediate result

$$\left\|\sup_{\theta\in\Theta_0}\left|h_{a,t}(\theta)-\tilde{h}_{a,t}(\theta)\right|\right\|_2 \le CU_{2,t}. \tag{34}$$

Now, using the equality $\frac{x}{y}-\frac{\tilde{x}}{\tilde{y}} = \frac{x(\tilde{y}-y)}{y\tilde{y}}+\frac{x-\tilde{x}}{\tilde{y}}$,

$$\left|\frac{h_{a,t}(\theta)}{h_t(\theta)}-\frac{\tilde{h}_{a,t}(\theta)}{\tilde{h}_t(\theta)}\right| \le \underline{\omega}^{-1}\left|\frac{h_{a,t}(\theta)}{h_t(\theta)}\right|\left|h_t(\theta)-\tilde{h}_t(\theta)\right| + \underline{\omega}^{-1}\left|h_{a,t}(\theta)-\tilde{h}_{a,t}(\theta)\right|,$$

and the desired result follows from part (iii), (34), and the result $\left\|\sup_{\theta\in\Theta_0}\left|\frac{h_{a,t}(\theta)}{h_t(\theta)}\right|\right\|_2 < \infty$ obtained in the proof of Lemma 6.

(viii) First consider the difference $|u_{b,t}(\theta)-\tilde{u}_{b,t}(\theta)|$. From the expressions of $u_{b_q,t}(\theta)$ and $\tilde{u}_{b_q,t}(\theta)$ in (28) and (27) we obtain

$$\begin{aligned}\left|u_{b_q,t}(\theta)-\tilde{u}_{b_q,t}(\theta)\right| &= \left|\sum_{j=0}^{\infty}\psi_j^{(b)}u_{t+q+j}(\theta)-\sum_{j=0}^{T-t}\psi_j^{(b)}u_{t+q+j}(\theta)+\sum_{j=0}^{T-t}\psi_j^{(b)}u_{t+q+j}(\theta)-\sum_{j=0}^{T-t}\psi_j^{(b)}\tilde{u}_{t+q+j}(\theta)\right| \\ &\le \sum_{j=T+1-t}^{\infty}\left|\psi_j^{(b)}u_{t+q+j}(\theta)\right| + \sum_{j=0}^{T-t-q}\left|\psi_j^{(b)}\right|\left|u_{t+q+j}(\theta)-\tilde{u}_{t+q+j}(\theta)\right| \\ &\quad + \sum_{j=T-t-q+1}^{T-t}\left|\psi_j^{(b)}\right|\left|u_{t+q+j}(\theta)-\tilde{u}_{t+q+j}(\theta)\right|.\end{aligned}$$

In the last term, $\tilde{u}_{T+1}(\theta)=\cdots=\tilde{u}_{T+Q}(\theta)=0$, so that

$$\begin{aligned}\left\|\sup_{\theta\in\Theta_0}\left|u_{b_q,t}(\theta)-\tilde{u}_{b_q,t}(\theta)\right|\right\|_4 &\le \sum_{j=T+1-t}^{\infty}\sup_{\theta\in\Theta_0}\left|\psi_j^{(b)}\right|\left\|\sup_{\theta\in\Theta_0}|u_{t+q+j}(\theta)|\right\|_4 \\ &\quad + \sum_{j=0}^{T-t-q}\sup_{\theta\in\Theta_0}\left|\psi_j^{(b)}\right|\left\|\sup_{\theta\in\Theta_0}|u_{t+q+j}(\theta)-\tilde{u}_{t+q+j}(\theta)|\right\|_4 \\ &\quad + \sum_{j=T-t-q+1}^{T-t}\sup_{\theta\in\Theta_0}\left|\psi_j^{(b)}\right|\left\|\sup_{\theta\in\Theta_0}|u_{t+q+j}(\theta)|\right\|_4.\end{aligned}$$

The first and third terms are dominated by $C_1\rho^{T+1-t}$ and $C_2\rho^{T+1-t}$, whereas the second one is dominated by

$$\sum_{j=0}^{T-t-q}C_3\rho^jC_4\rho^{T+1-(t+q+j)} \le C_5\,(T+1-t)\,\rho^{T+1-t}.$$

Therefore, we obtain the intermediate result

$$\left\|\sup_{\theta\in\Theta_0}|u_{b,t}(\theta)-\tilde{u}_{b,t}(\theta)|\right\|_4 \le C\,(T+1-t)\,\rho^{T+1-t} \tag{35}$$

for some finite C. Now, exactly as in the part (vi),

$$\left|\frac{u_{b,t}(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_{b,t}(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| \leq |u_{b,t}(\theta) - \tilde{u}_{b,t}(\theta)| \left|h_t^{-1/2}(\theta)\right| + |u_{b,t}(\theta) - \tilde{u}_{b,t}(\theta)| \left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right| + |u_{b,t}(\theta)| \left|h_t^{-1/2}(\theta) - \tilde{h}_t^{-1/2}(\theta)\right|,$$

and the result follows as in part (vi) except for using now (35) and the result $\left\|\sup_{\theta\in\Theta_0} |u_{b,t}(\theta)|\right\|_4 < \infty$ obtained in the proof of Lemma 6.

(ix) As in the beginning of part (vii), we can first derive the intermediate result

$$\left\|\sup_{\theta\in\Theta_0} \left|h_{b,t}(\theta) - \tilde{h}_{b,t}(\theta)\right|\right\|_2 \leq CU_{3,t}, \tag{36}$$

where the different upper bound results from the use of (35) instead of (33). Using (36) and the result $\left\|\sup_{\theta\in\Theta_0} \left|\frac{h_{b,t}(\theta)}{h_t(\theta)}\right|\right\|_2 < \infty$ obtained in the proof of Lemma 6, the proof of (ix) can be completed exactly as the proof of part (vii).

(x) First consider the difference $\left|h_{c,t}(\theta) - \tilde{h}_{c,t}(\theta)\right|$. The first component of $h_{c,t}(\theta) - \tilde{h}_{c,t}(\theta)$ is zero for all t, whereas the remaining components are $u_{t-r}^2(\theta) - \tilde{u}_{t-r}^2(\theta)$ $(r = 1, \ldots, R)$. Using part (ii), and arguments similar to those used in part (iii) for $t = 1, \ldots, R$, yield the intermediate result

$$\left\|\sup_{\theta\in\Theta_0} \left|h_{c,t}(\theta) - \tilde{h}_{c,t}(\theta)\right|\right\|_2 \leq CU_{2,t}. \tag{37}$$

The proof can be completed as the proof of part (vii), now making use of (37) and the result $\left\|\sup_{\theta\in\Theta_0} \left|\frac{h_{c,t}(\theta)}{h_t(\theta)}\right|\right\|_2 < \infty$.

(xi) From the definitions of $e_{x,t}(\theta)$ and $\tilde{e}_{x,t}(\theta)$ and Assumption 7a(i) one obtains

$$|e_{x,t}(\theta) - \tilde{e}_{x,t}(\theta)| \leq C\left[\left(1 + \left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)}\right|^{d_1}\right)\left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| + \left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right|^{d_2}\right].$$

Using Loève's c_r–inequality (see Davidson (1994), p. 140), the Cauchy-Schwarz inequality, part (v), the definition of Θ_c in Assumption 2 (ensuring $h_t(\theta) \geq \underline{\omega}$), and the fact that $\left\|\sup_{\theta\in\Theta_0} |u_t(\theta)|\right\|_4 < \infty$, we obtain the inequality $\left\|\sup_{\theta\in\Theta_0} |e_{x,t}(\theta) - \tilde{e}_{x,t}(\theta)|\right\|_{r_1} \leq CU_{2,t}$ for any $r_1 \leq \min\{2/3, 2/d_1, 4/3d_2\}$.

(xii) This can be proven exactly as part (xi), but relying now on Assumption 7a(ii).

(xiii) Making use of the definitions of $e_t(\theta)$ and $\tilde{e}_t(\theta)$ and the mean value theorem we obtain

$$e_t(\theta) - \tilde{e}_t(\theta) = \frac{f_{\eta,x}(w_t;\lambda)}{f_\eta(w_t;\lambda)}\left(\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right),$$

where, for each t, the intermediate point w_t is given by

$$w_t = \frac{u_t(\theta)}{h_t^{1/2}(\theta)} + \delta_t\left(\frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)} - \frac{u_t(\theta)}{h_t^{1/2}(\theta)}\right)$$

with each δ_t satisfying $0 \leq \delta_t \leq 1$. Adding and subtracting terms gives

$$e_t(\theta) - \tilde{e}_t(\theta) = \left(\frac{f_{\eta,x}(w_t;\lambda)}{f_\eta(w_t;\lambda)} - e_{x,t}(\theta)\right)\left(\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right) + e_{x,t}(\theta)\left(\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right)$$

and, using Assumption 7a(i) and the fact $0 \leq \delta_t \leq 1$,

$$\left|\frac{f_{\eta,x}(w_t;\lambda)}{f_\eta(w_t;\lambda)} - e_{x,t}(\theta)\right| \leq C\left[\left(1 + \left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)}\right|^{d_1}\right)\left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| + \left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right|^{d_2}\right]$$

By Loève's c_r–inequality, the Cauchy-Schwarz inequality, part (v), the definition of Θ_c in Assumption 2 (ensuring $h_t(\theta) \geq \underline{\omega}$), and the fact that $\left\|\sup_{\theta\in\Theta_0}|e_{x,t}(\theta)|\right\|_2 < \infty$ (established in the proof of Lemma 6), we obtain the inequality $\left\|\sup_{\theta\in\Theta_0}|e_t(\theta) - \tilde{e}_t(\theta)|\right\|_r \leq CU_{2,t}$ for some small enough r_3. ■

Proof of Lemma 7(ii). From the expressions of $l_{\theta,t}(\theta)$ and $\tilde{l}_{\theta,t}(\theta)$ (see the beginning of Appendix C and the beginning of this appendix) we find that

$$\begin{aligned}\left|l_{a,t}(\theta) - \tilde{l}_{a,t}(\theta)\right| \leq\ & \left|e_{x,t}(\theta)\frac{u_{a,t}(\theta)}{h_t^{1/2}(\theta)} - \tilde{e}_{x,t}(\theta)\frac{\tilde{u}_{a,t}(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| \\ & + \frac{1}{2}\left|\frac{h_{a,t}(\theta)}{h_t(\theta)}\left[e_{x,t}(\theta)\frac{u_t(\theta)}{h_t^{1/2}(\theta)} + 1\right] - \frac{\tilde{h}_{a,t}(\theta)}{\tilde{h}_t(\theta)}\left[\tilde{e}_{x,t}(\theta)\frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)} + 1\right]\right|.\end{aligned}$$

Using the inequality $|xy - \tilde{x}\tilde{y}| \leq |x - \tilde{x}|\,|y| + |x - \tilde{x}|\,|y - \tilde{y}| + |x|\,|y - \tilde{y}|$ three times on the majorant side, the following upper bound can be obtained for $\left|l_{a,t}(\theta) - \tilde{l}_{a,t}(\theta)\right|$:

$$\begin{aligned}&|e_{x,t}(\theta) - \tilde{e}_{x,t}(\theta)|\left|\frac{u_{a,t}(\theta)}{h_t^{1/2}(\theta)}\right| + |e_{x,t}(\theta) - \tilde{e}_{x,t}(\theta)|\left|\frac{u_{a,t}(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_{a,t}(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| \\ &+ |e_{x,t}(\theta)|\left|\frac{u_{a,t}(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_{a,t}(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| + \frac{1}{2}\left|\frac{h_{a,t}(\theta)}{h_t(\theta)} - \frac{\tilde{h}_{a,t}(\theta)}{\tilde{h}_t(\theta)}\right|\left|e_{x,t}(\theta)\frac{u_t(\theta)}{h_t^{1/2}(\theta)} + 1\right| \\ &+ \left(\frac{1}{2}\left|\frac{h_{a,t}(\theta)}{h_t(\theta)} - \frac{\tilde{h}_{a,t}(\theta)}{\tilde{h}_t(\theta)}\right| + \frac{1}{2}\left|\frac{h_{a,t}(\theta)}{h_t(\theta)}\right|\right) \times \\ &\left(|e_{x,t}(\theta) - \tilde{e}_{x,t}(\theta)|\frac{|u_t(\theta)|}{h_t^{1/2}(\theta)} + |e_{x,t}(\theta) - \tilde{e}_{x,t}(\theta)|\left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right| + |e_{x,t}(\theta)|\left|\frac{u_t(\theta)}{h_t^{1/2}(\theta)} - \frac{\tilde{u}_t(\theta)}{\tilde{h}_t^{1/2}(\theta)}\right|\right)\end{aligned}$$

This upper bound can further be bounded by using Loève's c_r–inequality, Hölder's inequality, the norm inequality, the definition of Θ_c in Assumption 2 (ensuring $h_t(\theta), \tilde{h}_t(\theta) \geq \underline{\omega}$), Lemma E.1, and moment results for the quantities $\sup_{\theta\in\Theta_0}|u_t(\theta)|$, $\sup_{\theta\in\Theta_0}|e_{x,t}(\theta)|$, $\sup_{\theta\in\Theta_0}|u_{a,t}(\theta)|$, and $\sup_{\theta\in\Theta_0}\left|\frac{h_{a,t}(\theta)}{h_t(\theta)}\right|$ obtained in the proof of Lemma 6. To conclude, we get

$$\left\|\sup_{\theta\in\Theta_0}\left|l_{a,t}(\theta) - \tilde{l}_{a,t}(\theta)\right|\right\|_p \leq CU_{2,t}$$

for some finite constant C and a small enough positive exponent p. As in the proof of part (i), it now follows that $\lim_{T\to\infty}\sum_{t=1}^{T}\sup_{\theta\in\Theta_0}\left|l_{a,t}(\theta)-\tilde{l}_{a,t}(\theta)\right|<\infty$ a.s., implying

$$T^{1/2}\sup_{\theta\in\Theta_0}\left|L_{a,T}\left(\theta\right)-\tilde{L}_{a,T}\left(\theta\right)\right|\leq T^{-1/2}\sum_{t=1}^{T}\sup_{\theta\in\Theta_0}\left|l_{a,t}(\theta)-\tilde{l}_{a,t}(\theta)\right|\to 0 \text{ a.s. as } T\to\infty.$$

Now consider $\left|l_{b,t}(\theta)-\tilde{l}_{b,t}(\theta)\right|$. Steps similar to those used above lead to the result

$$\left\|\sup_{\theta\in\Theta_0}\left|l_{b,t}(\theta)-\tilde{l}_{b,t}(\theta)\right|\right\|_p\leq C \text{ for } t=1,\ldots,R \text{ and } \leq C\left(T+1-t\right)\rho^{T+1-t} \text{ for } t=R+1,\ldots,T$$

with some positive p and sufficiently large T (in this case, T needs to be large enough to ensure that the constant upper bound C suffices for $t=1,\ldots,R$; see, e.g., the upper bound in Lemma E.1(viii)). As above, we obtain the results $\lim_{T\to\infty}\sum_{t=1}^{T}\sup_{\theta\in\Theta_0}\left|l_{b,t}(\theta)-\tilde{l}_{b,t}(\theta)\right|<\infty$ a.s. and

$$T^{1/2}\sup_{\theta\in\Theta_0}\left|L_{b,T}\left(\theta\right)-\tilde{L}_{b,T}\left(\theta\right)\right|\leq T^{-1/2}\sum_{t=1}^{T}\sup_{\theta\in\Theta_0}\left|l_{b,t}(\theta)-\tilde{l}_{b,t}(\theta)\right|\to 0 \text{ a.s. as } T\to\infty.$$

The corresponding results for $|l_{c,t}(\theta)-\tilde{l}_{c,t}(\theta)|$ and $|l_{d,t}(\theta)-\tilde{l}_{d,t}(\theta)|$ can be derived in a similar manner, so details are omitted. Thus, we can conclude that $T^{1/2}\sup_{\theta\in\Theta_0}\left|L_{\theta,T}\left(\theta\right)-\tilde{L}_{\theta,T}\left(\theta\right)\right|\to 0$ a.s. as $T\to\infty$ as desired. ■

Proof of Theorem 1, remaining details. We first prove claims (a) and (b). Consider S_1. Because the $l_{\theta,t}\left(\theta_0\right)$ form a stationary ergodic sequence with $E[l_{\theta,t}\left(\theta_0\right)]=0$, the ergodic theorem yields $L_{\theta,T}\left(\theta_0\right)=T^{-1}\sum_{t=1}^{T}l_{\theta,t}\left(\theta_0\right)\to 0$ a.s. as $T\to\infty$. Thus for any fixed ϵ, $\sup_{\theta\in\Theta_\epsilon}S_1\to 0$ a.s. as $T\to\infty$. Concerning S_2, because $\Theta_\varepsilon\subset\Theta_0$, we have $\theta_\bullet\in\Theta_0$. By Lemma 6, $\sup_{\theta\in\Theta_0}\left|L_{\theta\theta,T}\left(\theta\right)-\mathcal{J}\left(\theta\right)\right|\to 0$ a.s., and therefore $\sup_{\theta\in\Theta_\epsilon}S_2\to 0$ a.s. as $T\to\infty$.

Now consider claim (b) and first S_4. By Lemmas 2 and 5, the matrix $\mathcal{J}\left(\theta_0\right)$ is negative definite. Therefore we can write $\mathcal{J}\left(\theta_0\right)$ as $\mathbf{A}_{\mathcal{J}}'\mathbf{D}_{\mathcal{J}}\mathbf{A}_{\mathcal{J}}$, where $\mathbf{A}_{\mathcal{J}}$ is orthogonal and $\mathbf{D}_{\mathcal{J}}$ contains the eigenvalues of $\mathcal{J}\left(\theta_0\right)$, say $\lambda_1,\ldots,\lambda_{\dim(\theta)}$, which are all negative. Set $\delta=\frac{1}{4}\min\left\{\left|\lambda_1\right|,\ldots,\left|\lambda_{\dim(\theta)}\right|\right\}>0$. Therefore for all $\theta\in\Theta_\epsilon$,

$$\begin{aligned}\frac{1}{2}\left(\theta-\theta_0\right)'\mathcal{J}\left(\theta_0\right)\left(\theta-\theta_0\right) &= \frac{1}{2}\left[\mathbf{A}_{\mathcal{J}}\left(\theta-\theta_0\right)\right]'\mathbf{D}_{\mathcal{J}}\left[\mathbf{A}_{\mathcal{J}}\left(\theta-\theta_0\right)\right]=\frac{1}{2}\sum_{i=1}^{\dim(\theta)}\lambda_i\left[\mathbf{A}_{\mathcal{J}}\left(\theta-\theta_0\right)\right]_i^2\\ &\leq -2\delta\sum_{i=1}^{\dim(\theta)}\left[\mathbf{A}_{\mathcal{J}}\left(\theta-\theta_0\right)\right]_i^2=-2\delta\left|\theta-\theta_0\right|^2=-2\delta\epsilon^2,\end{aligned}$$

where the second last equality holds due to the orthogonality of $\mathbf{A}_{\mathcal{J}}$.

Finally, consider S_3. By Lemma 6, $\mathcal{J}\left(\theta\right)$ is continuous at θ_0. Therefore, we can find an $\tilde{\epsilon}$ such that $\left|\mathcal{J}\left(\theta\right)-\mathcal{J}\left(\theta_0\right)\right|\leq 2\delta/\dim\left(\theta\right)^2$ for all $\left|\theta-\theta_0\right|\leq\tilde{\epsilon}$. Supposing $\epsilon\leq\tilde{\epsilon}$, we thus have $\left|\mathcal{J}\left(\theta_\bullet\right)-\mathcal{J}\left(\theta_0\right)\right|\leq$

$2\delta/\dim(\theta)^2$, and also $[\mathcal{J}(\theta_\bullet) - \mathcal{J}(\theta_0)]_{i,j} \leq 2\delta/\dim(\theta)^2$ elementwise. Therefore for all $\theta \in \Theta_\epsilon$,

$$\begin{aligned}\frac{1}{2}(\theta-\theta_0)'[\mathcal{J}(\theta_\bullet) - \mathcal{J}(\theta_0)](\theta-\theta_0) &= \frac{1}{2}\sum_{i=1}^{\dim(\theta)}\sum_{j=1}^{\dim(\theta)}[\mathcal{J}(\theta_\bullet) - \mathcal{J}(\theta_0)]_{i,j}(\theta-\theta_0)_i(\theta-\theta_0)_j \\ &\leq \frac{\delta}{\dim(\theta)^2}\sum_{i=1}^{\dim(\theta)}\sum_{j=1}^{\dim(\theta)}(\theta-\theta_0)_i(\theta-\theta_0)_j \leq \delta\epsilon^2.\end{aligned}$$

Hence, claim (b) holds.

Next we prove that under Assumptions 1–7, $\sup_{\theta\in\Theta_0}\left|L_{\theta\theta,T}(\theta) - \tilde{L}_{\theta\theta,T}(\theta)\right| \to 0$ a.s. as $T \to \infty$. The method of proof is analogous to that used in the proofs of Lemmas E.1 and 7. For the sake of brevity, we only provide an outline of the required steps. As seen in the calculations at the beginning of this appendix, the expressions for $\tilde{l}_{\theta,t}(\theta)$ and its components are completely analogous to those of $l_{\theta,t}(\theta)$ and its corresponding components. Without presenting the formulas, we note that this is also the case for the second partial derivatives. Now, as in the proof of Lemma 7, we need to investigate the term $\left\|\sup_{\theta\in\Theta_0}|l_{\theta\theta,t}(\theta) - \tilde{l}_{\theta\theta,t}(\theta)|\right\|_p$ (for some $p > 0$) and show that it has a suitable upper bound. Like in the proof of Lemma 7, this requires us to derive results similar to those in Lemma E.1. An inspection of the expression of $l_{\theta\theta,t}(\theta)$ given at the beginning of Appendix D reveals that, in addition to the results of Lemma E.1, we need similar results for the differences of

$$e_{xx,t}(\theta),\quad e_{\lambda x,t}(\theta),\quad e_{\lambda\lambda,t}(\theta),\quad \mathsf{E}_{1,t}(\theta),\quad \mathsf{E}_{2,t}(\theta),\quad \mathsf{E}_{3,t}(\theta),\quad \frac{u_{aa,t}(\theta)}{h_t^{1/2}(\theta)},\quad \frac{u_{ba,t}(\theta)}{h_t^{1/2}(\theta)},\quad \frac{u_{bb,t}(\theta)}{h_t^{1/2}(\theta)},\quad \frac{h_{\theta_{abc}\theta_{abc},t}(\theta)}{h_t(\theta)},$$

and their feasible counterparts. First consider the three terms involving the second partial derivatives of $u_t(\theta)$. Their expressions derived at the beginning of Appendix D suggest that a technique similar to that used in parts (vi) and (viii) of the proof of Lemma E.1 applies also here and yields the needed results (without any additional assumptions). As for the second partial derivatives of $h_t(\theta)$, their expressions in Appendix D reveal that the necessary results will follow from those derived for the second partial derivatives of $u_t(\theta)$ and from those obtained in Lemma E.1. Finally, the expressions of $e_{xx,t}(\theta)$, $e_{\lambda x,t}(\theta)$, $e_{\lambda\lambda,t}(\theta)$, $\mathsf{E}_{1,t}(\theta)$, $\mathsf{E}_{2,t}(\theta)$, and $\mathsf{E}_{3,t}(\theta)$ show that, in addition to the results of Lemma E.1, all that is needed are results for the differences involving

$$\frac{f_{\eta,xx}(h_t^{-1/2}(\theta)u_t(\theta);\lambda)}{f_\eta(h_t^{-1/2}(\theta)u_t(\theta);\lambda)},\quad \frac{f_{\eta,\lambda x}(h_t^{-1/2}(\theta)u_t(\theta);\lambda)}{f_\eta(h_t^{-1/2}(\theta)u_t(\theta);\lambda)},\quad \frac{f_{\eta,\lambda\lambda}(h_t^{-1/2}(\theta)u_t(\theta);\lambda)}{f_\eta(h_t^{-1/2}(\theta)u_t(\theta);\lambda)},$$

and their feasible counterparts. This in turn follows from Assumption 7(b) exactly as in the proof of Lemma E.1(xi). ■

Additional References

Aliprantis, C. D., and O. Burkinshaw (1998): *Principles of Real Analysis*, 3rd ed., Academic Press, San Diego.