Supplementary material to "Testing for a Unit Root in Noncausal Autoregressive Models" by Pentti Saikkonen and Rickard Sandberg, Journal of Time Series Analysis 37(2016), pp. 99–125.

Details for establishing (25) in Appendix C

This section contains supplementary material for establishing (25) in Appendix C. Unless otherwise stated, $\theta \in N_{T,c}$ will be assumed. We only establish (25) for $\kappa = \phi, \varphi$. The other cases can be handled with similar arguments.

Case $\kappa = \phi$. First note that (see Appendix A)

$$
\begin{aligned}
T^{-2}\sum_{t=r+1}^{T-s}\left[g_{\phi\phi,t}(\theta) - g_{\phi\phi,t}(\theta_0)\right] &= T^{-2}\sum_{t=r+1}^{T-s}\left[\sigma^{-2}e_{xx,t}(\theta)\,u_{t-1}^2(\varphi) - \sigma_0^{-2}e_{xx,t}(\theta_0)\,u_{t-1}^2(\varphi_0)\right] \\
&= T^{-2}\sum_{t=r+1}^{T-s}\sigma_0^{-2}e_{xx,t}(\theta_0)\left[u_{t-1}^2(\varphi) - u_{t-1}^2(\varphi_0)\right] \\
&\quad + T^{-2}\sum_{t=r+1}^{T-s}\left[\sigma^{-2}e_{xx,t}(\theta) - \sigma_0^{-2}e_{xx,t}(\theta_0)\right]u_{t-1}^2(\varphi) \\
&\overset{def}{=} A_T^{(1)}(\theta) + A_T^{(2)}(\theta).
\end{aligned}
$$

Now consider $A_T^{(1)}(\theta)$ and note that

$$
u_{t-1}^2(\varphi) - u_{t-1}^2(\varphi_0) = 2u_{t-1}(\varphi_0)\left(u_{t-1}(\varphi) - u_{t-1}(\varphi_0)\right) + \left(u_{t-1}(\varphi) - u_{t-1}(\varphi_0)\right)^2,
$$

where

$$
u_{t-1}(\varphi) - u_{t-1}(\varphi_0) = \varphi\left(B^{-1}\right)y_t - \varphi_0\left(B^{-1}\right)y_t = Y_{1,t+1}'(\varphi - \varphi_0)
$$

with $Y_{1,t+1} = [y_{t+1} \ \cdots \ y_{t+s}]'$. Thus,

$$
\begin{aligned}
\sup_{\theta\in N_{T,c}}\left|A_T^{(1)}(\theta)\right| &\le \sup_{\theta\in N_{T,c}} T^{-2}\sum_{t=r+1}^{T-s}\left|\sigma_0^{-2}e_{xx,t}(\theta_0)\right|\left|u_{t-1}^2(\varphi) - u_{t-1}^2(\varphi_0)\right| \\
&\le \frac{2}{\sigma_0^2}\sup_{\theta\in N_{T,c}} T^{-2}\sum_{t=r+1}^{T-s}\left|e_{xx,t}(\theta_0)\right|\left|u_{t-1}(\varphi_0)\right|\left\|Y_{1,t+1}\right\|\left\|\varphi - \varphi_0\right\| \\
&\quad + \frac{1}{\sigma_0^2}\sup_{\theta\in N_{T,c}} T^{-2}\sum_{t=r+1}^{T-s}\left|e_{xx,t}(\theta_0)\right|\left\|Y_{1,t+1}\right\|^2\left\|\varphi - \varphi_0\right\|^2 \\
&\overset{def}{=} A_T^{(1,1)} + A_T^{(1,2)}.
\end{aligned}
$$

As a functional central limit theorem applies to $T^{-1/2}y_{[T\cdot]}$, we have $\max_{1\le t\le T}\left|T^{-1/2}y_t\right| = O_p(1)$. Thus, $\max_{1\le t\le T}\left\|T^{-1/2}Y_{1,t+1}\right\| = O_p(1)$ and, as $u_t(\varphi_0) = y_t - \varphi_{0,1}y_{t+1} - \cdots - \varphi_{0,s}y_{t+s}$, also $\max_{1\le t\le T}\left|T^{-1/2}u_t(\varphi_0)\right| = O_p(1)$. Because $\sup_{\theta\in N_{T,c}}\|\varphi - \varphi_0\| \le c/T^{1/2}$ we get, for all c,

$$
A_T^{(1,1)} \le \frac{2c}{\sigma_0^2}\max_{1\le t\le T}\left|T^{-1/2}u_t(\varphi_0)\right|\max_{1\le t\le T}\left\|T^{-1/2}Y_{1,t+1}\right\| T^{-3/2}\sum_{t=r+1}^{T-s}\left|e_{xx,t}(\theta_0)\right| = o_p(1).
$$

Here the equality follows because $|e_{xx,t}(\theta_0)|$ has finite expectation by Assumptions 3(i) and 4. Similar arguments also yield $A_T^{(1,2)} = o_p(1)$ so that altogether we have shown that

$$
\sup_{\theta\in N_{T,c}}\left|A_T^{(1)}(\theta)\right| = o_p(1).
$$

To obtain a similar result for $A_T^{(2)}(\theta)$ note first that, by the definition of $u_t(\varphi)$, $\max_{1\le t\le T}\left|T^{-1/2}u_t(\varphi)\right| \le C_1 \max_{1\le t\le T+s}\left|T^{-1/2}y_t\right| = O_p(1)$ for a finite C_1 independent of φ. Thus,

$$\sup_{\theta\in N_{T,c}}\left|A_T^{(2)}(\theta)\right| \le C_1^2 \max_{1\le t\le T}\left(T^{-1}y_t^2\right)\sup_{\theta\in N_{T,c}} T^{-1}\sum_{t=r+1}^{T-s}\left|\sigma^{-2}e_{xx,t}(\theta)-\sigma_0^{-2}e_{xx,t}(\theta_0)\right|,$$

so that it suffices to consider

$$\begin{aligned}\sup_{\theta\in N_{T,c}} T^{-1}\sum_{t=r+1}^{T-s}\left|\sigma^{-2}e_{xx,t}(\theta)-\sigma_0^{-2}e_{xx,t}(\theta_0)\right| &\le \sup_{\theta\in N_{T,c}}\left|\sigma^{-2}-\sigma_0^{-2}\right|T^{-1}\sum_{t=r+1}^{T-s}\left|e_{xx,t}(\theta_0)\right| \\ &\quad+\sup_{\theta\in N_{T,c}}\sigma^{-2}T^{-1}\sum_{t=r+1}^{T-s}\left|e_{xx,t}(\theta)-e_{xx,t}(\theta_0)\right| \\ &\overset{def}{=} A_T^{(2,1)}+A_T^{(2,2)}.\end{aligned}\tag{26}$$

We need to establish that $A_T^{(2,i)} = o_p(1)$, $i=1,2$. As $\sup_{\theta\in N_{T,c}}\left|\sigma^{-2}-\sigma_0^{-2}\right| = o(1)$, $A_T^{(2,1)} = o_p(1)$ immediately follows from the aforementioned fact that $E\left[\left|e_{xx,t}(\theta_0)\right|\right] < \infty$. A proof for $A_T^{(2,2)} = o_p(1)$ is obtained by using the inequality (24) and arguments similar to those above. To provide an idea of the needed arguments, first notice that (see Appendix A)

$$\epsilon_t(\theta) = \Delta u_t(\varphi) - \sum_{j=1}^{r-1}\pi_j\Delta u_{t-j}(\varphi) - \phi u_{t-1}(\varphi) \overset{def}{=} \tilde\epsilon_t(\theta) - \phi u_{t-1}(\varphi),$$

where $u_t(\varphi) = y_t - \varphi_1 y_{t+1} - \cdots - \varphi_s y_{t+s}$ with $\varphi_1,...,\varphi_s$ belonging to a bounded set. Thus, as $|\phi| \le c/T$ for $\theta\in N_{T,c}$, $\max_{1\le t\le T}|\phi u_t(\varphi)| = O_p\left(T^{-1/2}\right)$ holds uniformly in $N_{T,c}$ and

$$\left|\sigma^{-1}\epsilon_t(\theta)-\sigma_0^{-1}\epsilon_t(\theta_0)\right| \le \sigma^{-1}\left|\tilde\epsilon_t(\theta)-\epsilon_t(\theta_0)\right| + \left|\sigma^{-1}-\sigma_0^{-1}\right|\left|\epsilon_t(\theta_0)\right| + O_p\left(T^{-1/2}\right),$$

where the term $O_p\left(T^{-1/2}\right)$ is uniform over $1\le t\le T$ and $\theta\in N_{T,c}$. As $|a_1+a_2|^d \le M_d\left(|a_1|^d+|a_2|^d\right)$, $M_d<\infty$, for any real numbers a_1, a_2 and $d>0$, the desired result $A_T^{(2,2)} = o_p(1)$ can be obtained by applying the inequality (24) with the term $\left|\sigma^{-1}\epsilon_t(\theta)-\sigma_0^{-1}\epsilon_t(\theta_0)\right|$ on the right hand side replaced by $\sigma^{-1}\left|\tilde\epsilon_t(\theta)-\epsilon_t(\theta_0)\right| + \left|\sigma^{-1}-\sigma_0^{-1}\right|\left|\epsilon_t(\theta_0)\right|$. As $E\left[\left|\epsilon_t(\theta_0)\right|\right] = E\left[|\epsilon_t|\right] < \infty$, arguments similar to those used for the first term on the right hand side of (26) show that here the contribution of latter term is of order $o_p(1)$ so that we can replace $\left|\sigma^{-1}\epsilon_t(\theta)-\sigma_0^{-1}\epsilon_t(\theta_0)\right|$ on the right hand side of (24) by $\sigma^{-1}\left|\tilde\epsilon_t(\theta)-\epsilon_t(\theta_0)\right|$ and obtain

$$A_T^{(2,2)} \le C\sup_{\theta\in N_{T,c}}\sigma^{-2}T^{-1}\sum_{t=r+1}^{T-s}\left[\left(1+\left|\sigma_0^{-1}\epsilon_t(\theta_0)\right|^{d_1}\right)\sigma^{-1}\left|\tilde\epsilon_t(\theta)-\epsilon_t(\theta_0)\right| + \sigma^{-d_2}\left|\tilde\epsilon_t(\theta)-\epsilon_t(\theta_0)\right|^{d_2}\right] + o_p(1).$$

Here $\tilde\epsilon_t(\theta)-\epsilon_t(\theta_0) = -\sum_{j=1}^{r-1}(\pi_j-\pi_{0,j})\Delta u_{t-j}(\varphi)$ and $\sup_{\theta\in N_{T,c}}\|\pi-\pi_0\| \le c/T^{-1/2}$. Thus, as $\Delta u_t(\varphi) = \Delta y_t - \varphi_1\Delta y_{t+1} - \cdots - \varphi_s\Delta y_{t+s}$ with $\varphi_1,...,\varphi_s$ belonging to a bounded set, it follows that the majorant of $A_T^{(2,2)}$ is of order $o_p(1)$ (note that here use is also made of the assumptions $E[|\epsilon_t|^{1+d_1}] < \infty$ and $E[|\epsilon_t|^{d_2}] < \infty$ implied by their more stringent counterparts mentioned below (24)). Thus, we have established (25) with $\kappa = \phi$.

Case $\kappa = \varphi$. By the definition of $g_{\varphi\phi,t}(\theta)$ (see Appendix A)

$$
\begin{aligned}
& T^{-3/2} \sum_{t=r+1}^{T-s} \left[g_{\varphi\phi,t}(\theta) - g_{\varphi\phi,t}(\theta_0) \right] \\
&= T^{-3/2} \sum_{t=r+1}^{T-s} \left[\sigma^{-2} e_{xx,t}(\theta) u_{t-1}(\varphi) V_{t+1}(\phi, \pi) - \sigma_0^{-2} e_{xx,t}(\theta_0) u_{t-1}(\varphi_0) V_{t+1}(\phi_0, \pi_0) \right] \\
&\quad + T^{-3/2} \sum_{t=r+1}^{T-s} \left[\sigma^{-1} e_{x,t}(\theta) - \sigma_0^{-1} e_{x,t}(\theta_0) \right] Y_{1t} \\
&\overset{def}{=} B_T^{(1)}(\theta) + B_T^{(2)}(\theta).
\end{aligned}
$$

We need to show that $\sup_{\theta \in N_{T,c}} \left| B_T^{(i)}(\theta) \right| = o_p(1)$, $i = 1, 2$. First consider

$$
\begin{aligned}
\sup_{\theta \in N_{T,c}} \left| B_T^{(2)}(\theta) \right| &\le \sup_{\theta \in N_{T,c}} T^{-3/2} \sum_{t=r+1}^{T-s} \left| \sigma^{-1} e_{x,t}(\theta) - \sigma_0^{-1} e_{x,t}(\theta_0) \right| \left\| Y_{1,t+1} \right\| \\
&\le \max_{1 \le t \le T} \left\| T^{-1/2} Y_{1,t+1} \right\| \sup_{\theta \in N_{T,c}} T^{-1} \sum_{t=r+1}^{T-s} \left| \sigma^{-1} e_{x,t}(\theta) - \sigma_0^{-1} e_{x,t}(\theta_0) \right|,
\end{aligned}
$$

where $\max_{1 \le t \le T} \left\| T^{-1/2} Y_{1,t+1} \right\| = O_p(1)$, as already noted. That the supremum in the last expression is of order $o_p(1)$ can be established by using arguments similar to those used for $A_T^{(2)}(\theta)$ above, so details are omitted.

As for $B_T^{(1)}(\theta)$, we write

$$
\begin{aligned}
B_T^{(1)}(\theta) &= T^{-3/2} \sum_{t=r+1}^{T-s} \sigma_0^{-2} e_{xx,t}(\theta_0) \left[u_{t-1}(\varphi) V_{t+1}(\phi, \pi) - u_{t-1}(\varphi_0) V_{t+1}(\phi_0, \pi_0) \right] \\
&\quad + T^{-3/2} \sum_{t=r+1}^{T-s} \left[\sigma^{-2} e_{xx,t}(\theta) - \sigma_0^{-2} e_{xx,t}(\theta_0) \right] u_{t-1}(\varphi) V_{t+1}(\phi, \pi) \\
&\overset{def}{=} B_T^{(1,1)}(\theta) + B_T^{(1,2)}(\theta),
\end{aligned}
$$

and demonstrate that $\sup_{\theta \in N_{T,c}} \left| B_T^{(1,i)}(\theta) \right| = o_p(1)$, $i = 1, 2$.

First consider $B_T^{(1,1)}(\theta)$ and recall that $V_{t+1}(\phi, \pi) = (v_{t+1}(\phi, \pi), ..., v_{t+s}(\phi, \pi))$ with $v_t(\phi, \pi) = \phi(B) y_t$ or

$$
v_t(\phi, \pi) = \Delta y_t - \sum_{j=1}^{r-1} \pi_j \Delta y_{t-j} - \phi y_{t-1} \overset{def}{=} \tilde{v}_t(\pi) - \phi y_{t-1}
$$

(see (4)). Define $\tilde{V}_{t+1}(\pi) = (\tilde{v}_{t+1}(\pi), ..., \tilde{v}_{t+s}(\pi))$ so that $V_{t+1}(\phi, \pi) = \tilde{V}_{t+1}(\pi) - \phi y_{t-1} 1_s$ where $1_s = (1, ..., 1)$ $(s \times 1)$. Clearly, $\max_{1 \le t \le T} \left\| V_t(\phi, \pi) - \tilde{V}_t(\pi) \right\| = \sqrt{s} \max_{1 \le t \le T} |\phi y_t| = O_p\left(T^{-1/2}\right)$ uniformly in $N_{T,c}$, and $\tilde{V}_{t+1}(\pi_0) = V_{t+1}(\phi_0, \pi_0)$ is stationary. As we also have $E\left[|\epsilon_{xx,t}(\theta_0)| \right] < \infty$ and $\max_{1 \le t \le T} \left| T^{-1/2} u_t(\varphi) \right| = O_p(1)$ uniformly in $N_{T,c}$, we can replace $V_{t+1}(\phi, \pi)$ in $B_T^{(1,1)}(\theta)$ by

$\tilde{V}_{t+1}(\pi)$ and consider

$$
\begin{aligned}
\sup_{\theta\in N_{T,c}}\left\|\tilde{B}_T^{(1,1)}(\theta)\right\| &= \sup_{\theta\in N_{T,c}}\left\|T^{-3/2}\sum_{t=r+1}^{T-s}\sigma_0^{-2}e_{xx,t}(\theta_0)\left[u_{t-1}(\varphi)\tilde{V}_{t+1}(\pi)-u_{t-1}(\varphi_0)V_{t+1}(\phi_0,\pi_0)\right]\right\| \\
&\le \sup_{\theta\in N_{T,c}}\left\|T^{-3/2}\sum_{t=r+1}^{T-s}\sigma_0^{-2}e_{xx,t}(\theta_0)u_{t-1}(\varphi_0)\left[\tilde{V}_{t+1}(\pi)-V_{t+1}(\phi_0,\pi_0)\right]\right\| \\
&\quad+\sup_{\theta\in N_{T,c}}\left\|T^{-3/2}\sum_{t=r+1}^{T-s}\sigma_0^{-2}e_{xx,t}(\theta_0)\left[u_{t-1}(\varphi)-u_{t-1}(\varphi_0)\right]\tilde{V}_{t+1}(\pi)\right\| \\
&\overset{def}{=} \tilde{B}_T^{(1,3)}+\tilde{B}_T^{(1,4)}.
\end{aligned}
$$

First consider $\tilde{B}_T^{(1,4)}$ and recall that $u_{t-1}(\varphi)-u_{t-1}(\varphi_0)=Y_{1,t+1}'(\varphi-\varphi_0)$ and $\sup_{\theta\in N_{T,c}}\|\varphi-\varphi_0\|\le c/T^{1/2}$ (see the treatment of $A_T^{(1)}(\theta)$ in case $\kappa=\phi$). Thus,

$$
\begin{aligned}
\tilde{B}_T^{(1,4)} &\le \sigma_0^{-2}\sup_{\theta\in N_{T,c}}T^{-3/2}\sum_{t=r+1}^{T-s}|e_{xx,t}(\theta_0)|\,\|Y_{1,t+1}\|\,\|\varphi-\varphi_0\|\left\|\tilde{V}_{t+1}(\pi)\right\| \\
&\le c\sigma_0^{-2}\max_{1\le t\le T}\left\|T^{-1/2}Y_{1,t+1}\right\|\sup_{\theta\in N_{T,c}}T^{-3/2}\sum_{t=r+1}^{T-s}|e_{xx,t}(\theta_0)|\left\|\tilde{V}_{t+1}(\pi)\right\|,
\end{aligned}
$$

where $\max_{1\le t\le T}\left\|T^{-1/2}Y_{1,t+1}\right\|=O_p(1)$. That the last term is of order $o_p(1)$ now follows from the following two facts: (i) The components of $\tilde{V}_{t+1}(\pi)$ satisfy $|\tilde{v}_{t+i}(\pi)|\le C_2(|\Delta y_{t+i}|+\cdots+|\Delta y_{t+i-r+1}|)$ for some finite constant C_2 independent of π and (ii) using condition (A7) in L&S one can show that $E[|e_{xx,t}(\theta_0)\Delta y_{t+i-j}|]<\infty$ $(i=1,...,s,\ j=1,...,r-1)$.

Now consider $\tilde{B}_T^{(1,3)}$ for which we have

$$
\begin{aligned}
\tilde{B}_T^{(1,3)} &\le \sigma_0^{-2}\sup_{\theta\in N_{T,c}}T^{-3/2}\sum_{t=r+1}^{T-s}|e_{xx,t}(\theta_0)|\,|u_{t-1}(\varphi_0)|\left\|\tilde{V}_{t+1}(\pi)-V_{t+1}(\phi_0,\pi_0)\right\| \\
&\le \sigma_0^{-2}\max_{1\le t\le T}\left|T^{-1/2}u_t(\varphi_0)\right|\sup_{\theta\in N_{T,c}}T^{-1}\sum_{t=r+1}^{T-s}|e_{xx,t}(\theta_0)|\left\|\tilde{V}_{t+1}(\pi)-V_{t+1}(\phi_0,\pi_0)\right\|,
\end{aligned}
$$

where $\max_{1\le t\le T}\left|T^{-1/2}u_t(\varphi_0)\right|=O_p(1)$ as seen in the case $\kappa=\phi$. That the last term is of order $o_p(1)$ follows from the fact that the components of the vector $\tilde{V}_{t+1}(\pi)-V_{t+1}(\phi_0,\pi_0)$ are $\sum_{j=1}^{r-1}(\pi_j-\pi_{0,j})\Delta y_{t+i-j}$ with $\sup_{\theta\in N_{T,c}}\|\pi-\pi_0\|\le c/T^{1/2}$, and $E[|e_{xx,t}(\theta_0)\Delta y_{t+i-j}|]<\infty$ $(i=1,...,s,\ j=1,...,r-1)$, as noted above. Thus, we have established $\sup_{\theta\in N_{T,c}}\left|B_T^{(1,1)}(\theta)\right|=o_p(1)$, and we still need to obtain a similar result for $B_T^{(1,2)}(\theta)$.

By using arguments similar to those used for $B_T^{(1,1)}(\theta)$ it can first be seen that we can replace σ_0^{-2} in the definition of $B_T^{(1,2)}(\theta)$ by σ^{-2}, so that denoting the resulting quantity by $\tilde{B}_T^{(1,2)}(\theta)$ we can consider

$$
\begin{aligned}
\sup_{\theta\in N_{T,c}}\left\|\tilde{B}_T^{(1,2)}(\theta)\right\| &= \sup_{\theta\in N_{T,c}}\left\|\sigma^{-2}T^{-3/2}\sum_{t=r+1}^{T-s}\left[e_{xx,t}(\theta)-e_{xx,t}(\theta_0)\right]u_{t-1}(\varphi)V_{t+1}(\phi,\pi)\right\| \\
&\le C_3\max_{1\le t\le T}\left|T^{-1/2}y_t\right|\sup_{\theta\in N_{T,c}}T^{-1}\sum_{t=r+1}^{T-s}|e_{xx,t}(\theta)-e_{xx,t}(\theta_0)|\,\|V_{t+1}(\phi,\pi)\|,
\end{aligned}
$$

where $C_3 < \infty$ is independent of θ. Here the inequality is obtained by dominating $|u_{t-1}(\varphi)|$ in the same way as in handling $A_T^{(2)}(\theta)$. As $\max_{1\leq t\leq T}\left|T^{-1/2}y_t\right| = O_p(1)$ it suffices to show that the supremum in the last expression is of order $o_p(1)$. As seen above, $\max_{1\leq t\leq T}\left\|V_t(\phi,\pi) - \tilde{V}_t(\pi)\right\| = O_p\left(T^{-1/2}\right)$ which in conjunction with condition (A7) of L&S can be used to show that we can here replace $V_t(\phi,\pi)$ with the (stationary) variable $\tilde{V}_t(\pi)$ whose components, $\tilde{v}_{t+i}(\pi)$, satisfy $|\tilde{v}_{t+i}(\pi)| \leq C_2\left(|\Delta y_{t+i}| + \cdots + |\Delta y_{t+i-r+1}|\right)$ ($C_2 < \infty$, $i = 1, ..., s$). Thus, for $i \in \{1, ..., s\}$ it suffices to consider

$$\begin{aligned}
&\sup_{\theta\in N_{T,c}} T^{-1}\sum_{t=r+1}^{T-s}\left|e_{xx,t}(\theta) - e_{xx,t}(\theta_0)\right|\left|\Delta y_{t+i}\right| \\
\leq\ & C\sup_{\theta\in N_{T,c}} T^{-1}\sum_{t=r+1}^{T-s}\left[\left(1 + \left|\sigma_0^{-1}\epsilon_t(\theta_0)\right|^{d_1}\right)\left|\sigma^{-1}\epsilon_t(\theta) - \sigma_0^{-1}\epsilon_t(\theta_0)\right| + \left|\sigma^{-1}\epsilon_t(\theta) - \sigma_0^{-1}\epsilon_t(\theta_0)\right|^{d_2}\right]\left|\Delta y_{t+i}\right| \\
\leq\ & C\sup_{\theta\in N_{T,c}} \sigma^{-2}T^{-1}\sum_{t=r+1}^{T-s}\left[\left(1 + \left|\sigma_0^{-1}\epsilon_t(\theta_0)\right|^{d_1}\right)\sigma^{-1}\left|\tilde{\epsilon}_t(\theta) - \epsilon_t(\theta_0)\right| + \sigma^{-d_2}\left|\tilde{\epsilon}_t(\theta) - \epsilon_t(\theta_0)\right|^{d_2}\right]\left|\Delta y_{t+i}\right| + o_p(1).
\end{aligned}$$

Here the first inequality is based on (24) and the second one is obtained in the same way as its analog for $A_T^{(2,2)}$ in case $\kappa = \phi$. That the first term in the last expression is of order $o_p(1)$ can be seen in the same way as in the case of $A_T^{(2,2)}$ (see the end of case $\kappa = \phi$ and note that here we need the more stringent assumptions $E[|\epsilon_t|^{2+d_1}] < \infty$ and $E[|\epsilon_t|^{1+d_2}] < \infty$).