Supplementary Appendix to 'Subgeometrically ergodic autoregressions'

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Proof of Theorem 1. In the geometric case, the result of Theorem 1 is given in Meyn and Tweedie (2009, Thm 15.0.1). In the polynomial case, the result can be obtained by combining Theorem 2.8 of Douc et al. (2004) with the discussion in their Section 2.3 (see also Jarner and Roberts (2002)). In the subexponential case, the function ϕ is concave and increasing as long as v_0 is chosen large enough (cf. Douc et al. (2008, p. 243, the paragraph following Assumption 2)). Again, the result can be obtained by combining Theorem 2.8 of Douc et al. (2004) with the discussion in their Section 2.3; note also that the two functions $\phi(v) = c(v+v_0)/[\ln(v+v_0)]^{\alpha}$ and $\phi_0(v) = cv/[\ln(v)]^{\alpha}$ both lead to the same rate function $r_{\phi}(n)$ given in Douc et al. (2004, p. 1365, line 6). ■

Proof of Theorem 2. First note that from equation (16), Assumptions 1 and 2(a), and Theorem 2.2(ii) of Cline and Pu (1998) (see also Example 2.1 of that paper) it follows that the Markov chain $\boldsymbol{y}_t$ is a ψ-irreducible and aperiodic T-chain. Moreover, as in the proof of Lemma 1 of Lu (1998) it can be seen that ψ is the Lebesque measure and using Theorem 6.2.5 of Meyn and Tweedie (2009) we can conclude that all compact sets of $\mathcal{B}(\mathbb{R}^p)$ are petite (and in this case small, as shown by Theorem 5.5.7 of Meyn and Tweedie (2009)). The same also holds for the Markov chain y_t in the case $p = 1$.

In what follows, we first consider the case $p \geq 2$ and consider the case $p = 1$ at the end of the proof.

Part (i): In this case we have $\rho > \kappa_0$ and $b_3 = \kappa_0 \wedge (2-\rho) \in (0,1)$; for brevity, the notation b_3 will be used. The choice of b_1 and b_2 will be discussed later. We can make use of results in the proof of Theorem 3.3, part (i), in Douc et al. (2004, Sec. 3.3). Write the function $V(\boldsymbol{x})$ as

$$V(\boldsymbol{x}) = \tfrac{1}{2}\exp\{b_1 |z_1(\boldsymbol{x})|^{b_3}\} + \tfrac{1}{2}\exp\{b_2 \|\boldsymbol{z}_2(\boldsymbol{x})\|_*^{b_3}\} \stackrel{def}{=} V_1(z_1(\boldsymbol{x})) + V_2(\boldsymbol{z}_2(\boldsymbol{x})), \tag{30}$$

and consider $E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}]$, the conditional expectation in (9). Note that $\boldsymbol{z}(\boldsymbol{y}_1)$ appearing in $V(\boldsymbol{y}_1)$ can be expressed as (see (19))

$$\boldsymbol{z}(\boldsymbol{y}_1) = \begin{bmatrix} z_1(\boldsymbol{y}_1) \\ \boldsymbol{z}_2(\boldsymbol{y}_1) \end{bmatrix} = \boldsymbol{A}\boldsymbol{y}_1 = \begin{bmatrix} 0 & \boldsymbol{0}'_{p-1} \\ \boldsymbol{\iota}_{p-1} & \boldsymbol{\Pi}_1 \end{bmatrix} \begin{bmatrix} z_1(\boldsymbol{y}_0) \\ \boldsymbol{z}_2(\boldsymbol{y}_0) \end{bmatrix} + \overline{g}(\boldsymbol{y}_0)\boldsymbol{\iota}_p + \varepsilon_1 \boldsymbol{\iota}_p = \begin{bmatrix} \overline{g}(\boldsymbol{y}_0) + \varepsilon_1 \\ \boldsymbol{\Pi}_1 \boldsymbol{z}_2(\boldsymbol{y}_0) + z_1(\boldsymbol{y}_0)\boldsymbol{\iota}_{p-1} \end{bmatrix}.$$

In what follows, we usually drop the argument from $\boldsymbol{z}(\boldsymbol{x})$ and its components and write, for example, $\boldsymbol{z}_2$ instead of $\boldsymbol{z}_2(\boldsymbol{x})$. Now (dropping the argument from $\boldsymbol{z}(\boldsymbol{x})$)

$$\begin{aligned} E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] &= E\left[\tfrac{1}{2}\exp\left\{b_1 \left|\overline{g}(\boldsymbol{x}) + \varepsilon_1\right|^{b_3}\right\}\right] + \tfrac{1}{2}\exp\left\{b_2 \left\|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\right\|_*^{b_3}\right\} \\ &= E[V_1(\overline{g}(\boldsymbol{x}) + \varepsilon_1)] + V_2(\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}). \end{aligned}$$

Defining $V_\epsilon(\boldsymbol{x}) = \exp\{b_1 \left|\epsilon(\boldsymbol{x})\boldsymbol{x}\right|^{b_3}\}$ we bound the expectation on the right hand side as follows:

$$\begin{aligned} E\left[V_1\left(\overline{g}(\boldsymbol{x}) + \varepsilon_1\right)\right] &= E\left[\tfrac{1}{2}\exp\left\{b_1 \left|\overline{g}(\boldsymbol{x}) - g(z_1) + g(z_1) + \varepsilon_1\right|^{b_3}\right\}\right] \\ &\leq \exp\left\{b_1 \left|\overline{g}(\boldsymbol{x}) - g(z_1)\right|^{b_3}\right\} E\left[\tfrac{1}{2}\exp\left\{b_1 \left|g(z_1) + \varepsilon_1\right|^{b_3}\right\}\right] \\ &\leq V_\epsilon(\boldsymbol{x}) E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right], \end{aligned}$$

where the first inequality is due to the triangle inequality and the fact that $b_3 \in (0,1)$, and the second inequality follows from the definition of the function $\overline{g}$ and inequality (13) in Assumption 1(ii). Thus, we can bound the conditional expectation $E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right]$ as

$$E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq V_\epsilon(\boldsymbol{x}) E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right] + V_2\left(\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\right). \tag{31}$$

Step 1: Bounding $E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right]$ in (31). We first note that the arguments used by Douc et al. (2004) to obtain their inequality (3.14) can be used to justify that, for $|z_1| \geq M_0$,

$$V_1\left(g(z_1)\right) - V_1\left(z_1\right) \leq \left(-b_1 r b_3 \left|z_1\right|^{b_3 - \rho} + \tfrac{1}{2} b_1^2 r^2 b_3^2 \left|z_1\right|^{2(b_3-\rho)}\right) V_1(z_1).$$

Moreover, repeating the arguments in Douc et al. (2004) between their (3.15)–(3.19) it can be shown that, for $|z_1|$ large (which, due to our Assumption 1(ii), also implies that $|g(z_1)|$ is large) and some $c > 0$,

$$E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right] - V_1\left(g(z_1)\right) \leq \left\{\tfrac{1}{2} b_1^2 b_3^2 + c \left|z_1\right|^{-b_3}\right\} E[\varepsilon_1^2 V_1(\varepsilon_1)] \left|z_1\right|^{2b_3-2} V_1(z_1),$$

so that, for $|z_1|$ large,

$$E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right] - V_1\left(g(z_1)\right) \leq b_1^2 b_3^2 E[\varepsilon_1^2 V_1(\varepsilon_1)] \left|z_1\right|^{2b_3-2} V_1(z_1);$$

note that due to Assumption 2(a) and the choice of b_3, the condition $E\left[\left|\varepsilon_1\right|^2 V_1(\varepsilon_1)\right] < \infty$ can be achieved by choosing the value of b_1 small enough. From the above inequalities it follows that, for $|z_1|$ large,

$$E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right] \leq V_1(z_1) + k(z_1) V_1(z_1), \tag{32}$$

where

$$k(z_1) = -b_1 r b_3 \left|z_1\right|^{b_3-\rho} + \tfrac{1}{2} b_1^2 r^2 b_3^2 \left|z_1\right|^{2(b_3-\rho)} + b_1^2 b_3^2 E[\varepsilon_1^2 V_1(\varepsilon_1)] \left|z_1\right|^{2b_3-2}. \tag{33}$$

Next we obtain an upper bound for $k(z_1)$. Note that we necessarily have $b_3 - \rho < 0$ and $2b_3 - 2 \leq b_3 - \rho$ with equality if and only if $b_3 = 2 - \rho$ (these follow from $\rho > \kappa_0$ and

$b_3 = \kappa_0 \wedge (2-\rho)$). First consider the case $2b_3 - 2 = b_3 - \rho$ so that $b_3 = 2 - \rho$ and

$$k(z_1) = -\left(r - b_1 b_3 E[\varepsilon_1^2 V_1(\varepsilon_1)]\right) b_1 b_3 |z_1|^{b_3-\rho} + \tfrac{1}{2} b_1^2 r^2 b_3^2 |z_1|^{2(b_3-\rho)}.$$

As $b_3 - \rho < 0$, the inequality $\frac{1}{2} b_1 r^2 b_3 |z_1|^{b_3-\rho} \leq \frac{1}{2}\epsilon_1$ holds for all large enough $|z_1|$ and with $\epsilon_1 > 0$ which can be chosen as close to zero as desired. Moreover, as b_3 and $E[\varepsilon_1^2 V_1(\varepsilon_1)]$ are positive and here fixed, we can choose the value of b_1 small enough so that $b_1 b_3 E[\varepsilon_1^2 V_1(\varepsilon_1)] \leq \frac{1}{2}\epsilon_1$ holds. Hence, for $|z_1|$ large, $k(z_1) \leq -\left[r - \frac{1}{2}\epsilon_1 - \frac{1}{2}\epsilon_1\right] b_1 b_3 |z_1|^{b_3-\rho}$ and here ϵ_1 can be chosen small enough so that $r - \epsilon_1 > 0$ holds. Now consider the case $2b_3 - 2 < b_3 - \rho$ (so that $b_3 < 2 - \rho$). Write $k(z_1)$ as

$$k(z_1) = -(r - \tfrac{1}{2} b_1 r^2 b_3 |z_1|^{b_3-\rho} - b_1 b_3 E[\varepsilon_1^2 V_1(\varepsilon_1)] |z_1|^{(2b_3-2)-(b_3-\rho)}) b_1 b_3 |z_1|^{b_3-\rho}$$

and note that $\frac{1}{2} b_1 r^2 b_3 |z_1|^{b_3-\rho} + b_1 b_3 E[\varepsilon_1^2 V_1(\varepsilon_1)] |z_1|^{(2b_3-2)-(b_3-\rho)} \leq \epsilon_2$ holds with $0 < \epsilon_2 < r$ for all large enough $|z_1|$ so that the bound $k(z_1) \leq -(r-\epsilon_2) b_1 b_3 |z_1|^{b_3-\rho}$ is obtained. To combine the two cases, note that the arguments above hold if ϵ_1 and ϵ_2 are replaced with $\epsilon_3 = \epsilon_1 \wedge \epsilon_2$. Thus, defining the positive constant ω_1 as $\omega_1 = r - \epsilon_3$ we obtain, for $|z_1|$ large,

$$k(z_1) \leq -\omega_1 b_1 b_3 |z_1|^{b_3-\rho}$$

(cf. Douc et al. (2004, top of p. 1373)). Combining this with the inequality (32) we obtain

$$E[V_1(g(z_1) + \varepsilon_1)] \leq (1 - \omega_1 b_1 b_3 |z_1|^{b_3-\rho}) V_1(z_1).$$

Step 2: Bounding $V_\epsilon(\boldsymbol{x}) E[V_1(g(z_1)+\varepsilon_1)]$ in (31). Using the bound just obtained, bound the first term on the right hand side of (31) as

$$\begin{aligned} V_\epsilon(\boldsymbol{x}) E[V_1(g(z_1)+\varepsilon_1)] &\leq \exp\{b_1 |\epsilon(\boldsymbol{x})\boldsymbol{x}|^{b_3}\} (1 - \omega_1 b_1 b_3 |z_1|^{b_3-\rho}) V_1(z_1) \\ &= \tfrac{1}{2}(1 - \omega_1 b_1 b_3 |z_1|^{b_3-\rho}) \exp\{b_1 |z_1|^{b_3} + b_1 |\epsilon(\boldsymbol{x})\boldsymbol{x}|^{b_3}\}. \end{aligned}$$

For all $|z_1|$ large enough, $1 - \omega_1 b_1 b_3 |z_1|^{b_3-\rho} \in (0,1)$ and the same holds true for $k_1(z_1) \overset{def}{=} 1 - \frac{1}{2}\omega_1 b_1 b_3 |z_1|^{b_3-\rho}$. Using the inequality $(1-u)^\alpha \leq 1 - \alpha u$ $(0 \leq u, \alpha \leq 1)$ we thus have

$$1 - \omega_1 b_1 b_3 |z_1|^{b_3-\rho} = \left(1 - \omega_1 b_1 b_3 |z_1|^{b_3-\rho}\right)^{1/2} \left(1 - \omega_1 b_1 b_3 |z_1|^{b_3-\rho}\right)^{1/2} \leq k_1(z_1)^2.$$

Furthermore, as $\ln(k_1(z_1)) = \ln(1 - \frac{1}{2}\omega_1 b_1 b_3 |z_1|^{b_3-\rho}) \leq -\frac{1}{2}\omega_1 b_1 b_3 |z_1|^{b_3-\rho}$ it follows that $k_1(z_1) = \exp\{\ln(k_1(z_1))\} \leq \exp\{-\frac{1}{2}\omega_1 b_1 b_3 |z_1|^{b_3-\rho}\}$ and we can write

$$V_\epsilon(\boldsymbol{x}) E[V_1(g(z_1)+\varepsilon_1)] \leq \tfrac{1}{2} k_1(z_1) \exp\{\left(1 - \tfrac{1}{2}\omega_1 b_3 |z_1|^{-\rho}\right) b_1 |z_1|^{b_3} + b_1 |\epsilon(\boldsymbol{x})\boldsymbol{x}|^{b_3}\}.$$

Consider the argument of the exponential function on the right hand side of the above inequality. As $\boldsymbol{z} = (z_1, \boldsymbol{z}_2) = \boldsymbol{A}\boldsymbol{x}$, the equivalence of vector norms in $\mathbb{R}^p$ and straightforward calculations show that, for some $c_* > 0$,

$$|\epsilon(\boldsymbol{x})\boldsymbol{x}| = \left|\epsilon(\boldsymbol{x})\boldsymbol{A}^{-1}\boldsymbol{z}\right| \leq c_* |\epsilon(\boldsymbol{x})| |z_1| + c_* |\epsilon(\boldsymbol{x})| \|\boldsymbol{z}_2\|_* = |\epsilon_1(\boldsymbol{x})| |z_1| + |\epsilon_1(\boldsymbol{x})| \|\boldsymbol{z}_2\|_*,$$

where $\epsilon_1(\boldsymbol{x}) = c_*\epsilon(\boldsymbol{x})$. Hence, as Assumption 1(ii) holds with $d = \rho/b_3$, we have $|\epsilon_1(\boldsymbol{x})| = o(|\boldsymbol{x}|^{-\rho/b_3})$ and

$$|\epsilon(\boldsymbol{x})\boldsymbol{x}|^{b_3} \leq o(|\boldsymbol{x}|^{-\rho})\,|z_1|^{b_3} + o(|\boldsymbol{x}|^{-\rho})\,\|\boldsymbol{z}_2\|_*^{b_3},$$

so that, for all $|z_1|$ large (implying that $|\boldsymbol{z}|$ and hence that $|\boldsymbol{x}|$ is large[7]; see the discussion above Theorem 1),

$$\begin{aligned}\left(1 - \tfrac{1}{2}\omega_1 b_3\,|z_1|^{-\rho}\right) b_1\,|z_1|^{b_3} + b_1\,|\epsilon(\boldsymbol{x})\boldsymbol{x}|^{b_3} &\leq \left(1 - \tfrac{1}{2}\omega_1 b_3\,|z_1|^{-\rho} + o(|\boldsymbol{x}|^{-\rho})\right) b_1\,|z_1|^{b_3} + o(|\boldsymbol{x}|^{-\rho}) b_1\,\|\boldsymbol{z}_2\|_*^{b_3} \\ &\leq \left(1 - \omega_2 b_3\,|z_1|^{-\rho}\right) b_1\,|z_1|^{b_3} + o(|\boldsymbol{x}|^{-\rho}) b_1\,\|\boldsymbol{z}_2\|_*^{b_3},\end{aligned}$$

where $0 < \omega_2 < \frac{1}{2}\omega_1$. Thus, we can conclude that, for all $|z_1|$ large,

$$V_\epsilon(\boldsymbol{x})\,E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right] \leq \tfrac{1}{2}k_1(z_1)\exp\left\{(1 - \omega_2 b_3\,|z_1|^{-\rho}) b_1\,|z_1|^{b_3} + o(|\boldsymbol{x}|^{-\rho}) b_1\,\|\boldsymbol{z}_2\|_*^{b_3}\right\}.$$

Next define $\tau_1(z_1) = 1 - \omega_2 b_3\,|z_1|^{-\rho}$ and $\tau_2(z_1) = 1 - \tau_1(z_1)$, and note that $\tau_1(z_1) \in (0,1)$ for any $|z_1|$ large. By the preceding discussion, we then have, for all $|z_1|$ large,

$$\begin{aligned}V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right] &\leq \tfrac{1}{2}k_1(z_1)\exp\left\{\tau_1(z_1) b_1\,|z_1|^{b_3} + \tau_2(z_1)\tau_2(z_1)^{-1} o(|\boldsymbol{x}|^{-\rho}) b_1\,\|\boldsymbol{z}_2\|_*^{b_3}\right\} \\ &\leq \frac{\tau_1(z_1)}{2} k_1(z_1)\exp\left\{b_1\,|z_1|^{b_3}\right\} + \frac{\tau_2(z_1)}{2} k_1(z_1)\exp\left\{\tau_2(z_1)^{-1} o(|\boldsymbol{x}|^{-\rho}) b_1\,\|\boldsymbol{z}_2\|_*^{b_3}\right\} \\ &\leq \tfrac{1}{2}k_1(z_1)\exp\left\{b_1\,|z_1|^{b_3}\right\} + \tfrac{1}{2}\exp\left\{\tau_2(z_1)^{-1} o(|\boldsymbol{x}|^{-\rho})\,\|\boldsymbol{z}_2\|_*^{b_3}\right\} \\ &= k_1(z_1)V_1(z_1) + \tfrac{1}{2}\exp\left\{o(1)\,\|\boldsymbol{z}_2\|_*^{b_3}\right\}.\end{aligned}$$

Here the second inequality is justified by the convexity of the exponential function and the third one follows because $\tau_1(z_1) \in (0,1)$ and $k_1(z_1) \in (0,1)$ can be assumed. The last equality is due to the definition of V_1 and the definition of $\tau_2(z_1)$ which implies

$$\tau_2(z_1)^{-1} o(|\boldsymbol{x}|^{-\rho}) = (\omega_2 b_3)^{-1}\,|z_1|^{\rho}\,o(|\boldsymbol{x}|^{-\rho}) \leq c^{\rho}\,(\omega_2 b_3)^{-1}\,|\boldsymbol{x}|^{\rho}\,o(|\boldsymbol{x}|^{-\rho}) = o(1),$$

where the inequality holds because $|z_1| \leq |\boldsymbol{z}| \leq c\,|\boldsymbol{x}|$ (see footnote 7) and where $o(1) \to 0$ as $|\boldsymbol{x}| \to \infty$.

It will be convenient to modify the preceding upper bound of $V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right]$. To this end, denote $\alpha = \rho/b_3 - 1$ (> 0) and write $b_1\,|z_1|^{b_3-\rho} = b_1^{\rho/b_3}\left(b_1\,|z_1|^{b_3}\right)^{-\alpha} \geq b_1^{\rho/b_3}\left(1 + \ln V_1(z_1)\right)^{-\alpha}$ where the inequality is based on the definition of $V_1(z_1)$ (also note that $\ln(\frac{1}{2}) \approx -0.6931$). Thus, by the definition of $k_1(z_1)$ we have,

$$k_1(z_1) \leq 1 - \tfrac{1}{2}\omega_1 b_3 b_1^{\rho/b_3}\left(1 + \ln V_1(z_1)\right)^{-\alpha}.$$

Using this upper bound and the definition

$$\phi_1\left(V_1(z_1)\right) = \tfrac{1}{2}\omega_1 b_3 b_1^{\rho/b_3}\left(1 + \ln V_1(z_1)\right)^{-\alpha} V_1(z_1) \quad (> 0), \tag{34}$$

yields, for $|z_1|$ large and for a small enough choice of b_1, the following bound for the first term

[7] Due to the nonsingularity of the matrix $\boldsymbol{A}$, there exists a positive constant c such that $c^{-1}\,|\boldsymbol{z}| \leq |\boldsymbol{x}| \leq c\,|\boldsymbol{z}|$, so that $|\boldsymbol{x}| \to \infty$ if and only if $|\boldsymbol{z}| \to \infty$.

on the right hand side of (31):

$$V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1)+\varepsilon_1\right)\right] \le V_1(z_1) - \phi_1\left(V_1(z_1)\right) + \tfrac{1}{2}\exp\left\{o(1)\left\|\boldsymbol{z}_2\right\|_*^{b_3}\right\}.$$

To state this more formally, we can find $b_1 = \tilde{b}_1 < \beta_0$, and $M_1 \ge M_0$ such that the above inequality holds for $|z_1| > M_1$. Moreover, as in Douc et al. (2004, p. 1373) these choices can be done in such a way that, for some (finite) constant $\overline{M}_1$, and for all z_1,

$$V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1)+\varepsilon_1\right)\right] \le V_1(z_1) - \phi_1\left(V_1(z_1)\right) + \tfrac{1}{2}\exp\left\{o(1)\left\|\boldsymbol{z}_2\right\|_*^{b_3}\right\} + \overline{M}_1\mathbf{1}_{C_1}(z_1), \tag{35}$$

where $C_1 = \{z_1 \in \mathbb{R} : |z_1| \le M_1\}$.

Step 3: Bounding $V_2\left(\boldsymbol{\Pi}_1\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\right)$ in (31). Here we assume that the choice of b_1 is fixed to the value $\tilde{b}_1$ specified above. Recall that $V_2(\boldsymbol{\Pi}_1\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}) = \frac{1}{2}\exp\left\{b_2\left\|\boldsymbol{\Pi}_1\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\right\|_*^{b_3}\right\}$ and note that

$$b_2\left\|\boldsymbol{\Pi}_1\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\right\|_*^{b_3} \le b_2\left(\left\|\boldsymbol{\Pi}_1\boldsymbol{z}_2\right\|_* + \left\|z_1\boldsymbol{\iota}_{p-1}\right\|_*\right)^{b_3} \le b_2\eta^{b_3}\left\|\boldsymbol{z}_2\right\|_*^{b_3} + b_2\left\|\boldsymbol{\iota}_{p-1}\right\|_*^{b_3}|z_1|^{b_3},$$

where we have made use of the fact $b_3 \in (0,1)$ and Assumption 1(i) which implies that $\left\|\boldsymbol{\Pi}_1\right\|_* \le \eta$ for some $\eta < 1$ (see the discussion following equation (19)).

Let $\tau_1 \in (0,1)$ and $\tau_2 = 1 - \tau_1$ be such that $\tau_2 \in (\eta^{b_3}, 1)$, and denote $b_{2,1} = b_2\left\|\boldsymbol{\iota}_{p-1}\right\|_*^{b_3}/\tau_1$ and $b_{2,2} = b_2/\tau_2$. Then,

$$\begin{aligned}
V_2\left(\boldsymbol{\Pi}_1\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\right) &\le \tfrac{1}{2}\exp\left\{b_2\eta^{b_3}\left\|\boldsymbol{z}_2\right\|_*^{b_3} + b_2\left\|\boldsymbol{\iota}_{p-1}\right\|_*^{b_3}|z_1|^{b_3}\right\}\\
&= \tfrac{1}{2}\exp\left\{\tau_2 b_{2,2}\eta^{b_3}\left\|\boldsymbol{z}_2\right\|_*^{b_3} + \tau_1 b_{2,1}|z_1|^{b_3}\right\}\\
&\le \tfrac{\tau_1}{2}\exp\left\{b_{2,1}|z_1|^{b_3}\right\} + \tfrac{\tau_2}{2}\exp\left\{b_{2,2}\eta^{b_3}\left\|\boldsymbol{z}_2\right\|_*^{b_3}\right\}\\
&\le \tfrac{1}{2}\exp\left\{b_{2,1}|z_1|^{b_3}\right\} + \tfrac{1}{2}\exp\left\{b_{2,2}\eta^{b_3}\left\|\boldsymbol{z}_2\right\|_*^{b_3}\right\}\\
&\overset{def}{=} V_{2,1}(z_1) + V_{2,2}(\boldsymbol{z}_2),
\end{aligned}$$

where the second inequality is justified by the convexity of the exponential function. Now, as $\tau_2 \in (\eta^{b_3}, 1)$, we have $b_{2,2}\eta^{b_3} = b_2\eta^{b_3}/\tau_2 < b_2$, and we choose the value of b_2 so small that $b_{2,1} = b_2\left\|\boldsymbol{\iota}_{p-1}\right\|_*^{b_3}/\tau_1 < b_1 = \tilde{b}_1$ with $\tilde{b}_1$ as fixed above.

We next bound $V_{2,1}(z_1)$ and $V_{2,2}(\boldsymbol{z}_2)$. For the former, write $V_{2,1}(z_1) = \exp\left\{-\left(b_1 - b_{2,1}\right)|z_1|^{b_3}\right\}V_1(z_1)$ and use the facts $\ln V_1(z_1) = \ln(\frac{1}{2}) + b_1|z_1|^{b_3}$, $\alpha = \rho/b_3 - 1 > 0$, and $b_1 - b_{2,1} > 0$ to obtain

$$V_{2,1}(z_1) = \frac{\left(1+\ln V_1(z_1)\right)^{\alpha} b_3 b_1^{\rho/b_3}}{b_3 b_1^{\rho/b_3}\exp\left\{\left(b_1 - b_{2,1}\right)|z_1|^{b_3}\right\}}\left(1+\ln V_1(z_1)\right)^{-\alpha}V_1(z_1) \le \tfrac{1}{2}\epsilon_4 b_3 b_1^{\rho/b_3}\left(1+\ln V_1(z_1)\right)^{-\alpha}V_1(z_1),$$

where the inequality holds for any $\epsilon_4 > 0$ as long as $|z_1|$ is large enough. Using the definition of $\phi_1\left(V_1(z_1)\right)$ in (34) this implies a bound for $-\phi_1\left(V_1(z_1)\right) + V_{2,1}(z_1)$ which will be needed later:

$$\begin{aligned}
-\phi_1\left(V_1(z_1)\right) + V_{2,1}(z_1) &\le -\tfrac{1}{2}\omega_1 b_3 b_1^{\rho/b_3}\left(1+\ln V_1(z_1)\right)^{-\alpha}V_1(z_1)\\
&\quad + \tfrac{1}{2}\epsilon_4 b_3 b_1^{\rho/b_3}\left(1+\ln V_1(z_1)\right)^{-\alpha}V_1(z_1) + \overline{M}_1\mathbf{1}_{C_1}(z_1)\\
&= -\omega b_3 b_1^{\rho/b_3}\left(1+\ln V_1(z_1)\right)^{-\alpha}V_1(z_1) + \overline{M}_1\mathbf{1}_{C_1}(z_1),
\end{aligned} \tag{36}$$

where $\omega = \frac{1}{2}(\omega_1 - \epsilon_4)$ and, as $\omega_1 > 0$ holds for (fixed) $b_1 = \tilde{b}_1$, we can choose ϵ_4 so small that $\omega > 0$ holds. Note that here the last expression provides a bound for $-\phi_1(V_1(z_1)) + V_{2,1}(z_1)$ that holds for all z_1 (although this may require redefining the set C_1 and the value of the constant $\overline{M}_1$ which appear also in the upper bound obtained earlier for $E[V_1(g(z_1) + \varepsilon_1)]$). Denoting $\epsilon = \epsilon_3 + \epsilon_4$ and using the definition of ω_1 (given at the end of Step 1) we therefore have $\omega = \frac{1}{2}(r - \epsilon)$.

Now consider $V_{2,2}(\boldsymbol{z}_2) = \frac{1}{2}\exp\{b_{2,2}\eta^{b_3}\|\boldsymbol{z}_2\|_*^{b_3}\}$ and recall that $b_{2,2}\eta^{b_3} < b_2$. Using the definition $V_2(\boldsymbol{z}_2) = \frac{1}{2}\exp\{b_2\|\boldsymbol{z}_2\|_*^{b_3}\}$ we have, for some $\eta_2 \in (0,1)$ and $\|\boldsymbol{z}_2\|_*$ bounded away from zero,

$$V_{2,2}(\boldsymbol{z}_2) = V_2(\boldsymbol{z}_2)\frac{\exp\{b_{2,2}\eta^{b_3}\|\boldsymbol{z}_2\|_*^{b_3}\}}{\exp\{b_2\|\boldsymbol{z}_2\|_*^{b_3}\}} = V_2(\boldsymbol{z}_2)\exp\{-\left(b_2 - b_{2,2}\eta^{b_3}\right)\|\boldsymbol{z}_2\|_*^{b_3}\} \leq \eta_2 V_2(\boldsymbol{z}_2),$$

and furthermore

$$V_2\big(\boldsymbol{\Pi}_1\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\big) \leq V_{2,1}(z_1) + \eta_2 V_2(\boldsymbol{z}_2),$$

where the bound obtained above for $V_{2,1}(z_1)$ has been omitted but it will be used below.

Step 4: Bounding $E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}]$ in (31). Using (35) and the preceding inequality obtained for $V_2\big(\boldsymbol{\Pi}_1\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\big)$ we can now write

$$\begin{aligned} E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq V_1(z_1) - \phi_1(V_1(z_1)) + \tfrac{1}{2}\exp\big\{o(1)\|\boldsymbol{z}_2\|_*^{b_3}\big\} + \overline{M}_1\mathbf{1}_{C_1}(z_1) \\ + V_{2,1}(z_1) + V_2(\boldsymbol{z}_2) - (1-\eta_2)V_2(\boldsymbol{z}_2). \end{aligned}$$

As $|\boldsymbol{z}_2| \leq |\boldsymbol{z}| \leq c\,|\boldsymbol{x}|$ (see footnote 7), the term $o(1)$ on the right hand side converges to zero as $|\boldsymbol{z}_2| \to \infty$. Thus, as $V_2(\boldsymbol{z}_2) = \frac{1}{2}\exp\{b_2\|\boldsymbol{z}_2\|_*^{b_3}\}$, we have, for $|\boldsymbol{z}_2|$ large,

$$\begin{aligned} \tfrac{1}{2}\exp\big\{o(1)\|\boldsymbol{z}_2\|_*^{b_3}\big\} - (1-\eta_2)V_2(\boldsymbol{z}_2) &= \big[\exp\big\{[o(1) - b_2]\|\boldsymbol{z}_2\|_*^{b_3}\big\} - (1-\eta_2)\big]\tfrac{1}{2}\exp\big\{b_2\|\boldsymbol{z}_2\|_*^{b_3}\big\} \\ &\leq -\eta_3 V_2(\boldsymbol{z}_2), \end{aligned}$$

where $\eta_3 \in (0,1)$. Hence,

$$V_2(\boldsymbol{z}_2) + \tfrac{1}{2}\exp\big\{o(1)\|\boldsymbol{z}_2\|_*^{b_3}\big\} - (1-\eta_2)V_2(\boldsymbol{z}_2) \leq V_2(\boldsymbol{z}_2) - \eta_3 V_2(\boldsymbol{z}_2) + \overline{M}_2\mathbf{1}_{C_2}(\boldsymbol{z}_2),$$

where $C_2 = \{\boldsymbol{z}_2 \in \mathbb{R}^{p-1} : |\boldsymbol{z}_2| \leq M_2\}$ and M_2 and $\overline{M}_2$ are some finite constants. Using this inequality and the bound in (36) we can bound $E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}]$ as follows:

$$\begin{aligned} E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq V_1(z_1) - \omega b_3 b_1^{\rho/b_3}(1 + \ln V_1(z_1))^{-\alpha}V_1(z_1) \\ + V_2(\boldsymbol{z}_2) - \eta_3 V_2(\boldsymbol{z}_2) + 2\overline{M}_1\mathbf{1}_{C_1}(z_1) + \overline{M}_2\mathbf{1}_{C_2}(\boldsymbol{z}_2). \end{aligned}$$

We still need to modify the right hand side of the above inequality to a form assumed in Condition D, and for simplicity we write this inequality as

$$E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq V_1(z_1) - \omega b_3 b_1^{\rho/b_3}(1 + \ln V_1(z_1))^{-\alpha}V_1(z_1) + V_2(\boldsymbol{z}_2) - \eta_3 V_2(\boldsymbol{z}_2) + L,$$

where $L \geq 2\overline{M}_1 + \overline{M}_2$. Next note that $V(\boldsymbol{x}) \geq V_1(z_1) \geq 1/2$ (see (30)) so that

$$0 < (1 + \ln V(\boldsymbol{x}))^{-\alpha} \leq (1 + \ln V_1(z_1))^{-\alpha} \leq (1 + \ln(1/2))^{-\alpha}.$$

Using these inequalities twice and defining $c_\phi = \omega b_3 b_1^{\rho/b_3} \wedge \eta_3 \left(1 + \ln \frac{1}{2}\right)^\alpha$ (> 0) we have

$$\begin{aligned}
&- \omega b_3 b_1^{\rho/b_3} (1 + \ln V_1(z_1))^{-\alpha} V_1(z_1) - \eta_3 V_2(\boldsymbol{z}_2) \\
&\quad \leq -\omega b_3 b_1^{\rho/b_3} (1 + \ln V(\boldsymbol{x}))^{-\alpha} V_1(z_1) - \eta_3 (1 + \ln(1/2))^\alpha (1 + \ln V(\boldsymbol{x}))^{-\alpha} V_2(\boldsymbol{z}_2) \\
&\quad \leq -c_\phi (1 + \ln V(\boldsymbol{x}))^{-\alpha} V(\boldsymbol{x}).
\end{aligned}$$

Denoting $h(\boldsymbol{x}) = c_\phi (1 + \ln V(\boldsymbol{x}))^{-\alpha}$ we therefore obtain

$$E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq (1 - h(\boldsymbol{x}))\, V(\boldsymbol{x}) + L. \tag{37}$$

Because $V(\boldsymbol{x}) \geq 1$ and $-\alpha < 0$, we have $0 < h(\boldsymbol{x}) \leq c_\phi$ and $h(\boldsymbol{x}) \to 0$, as $|\boldsymbol{x}| \to \infty$. Thus, for all $|\boldsymbol{x}|$ large enough, $h(\boldsymbol{x}) \leq 1$, and therefore

$$\begin{aligned}
(1 - h(\boldsymbol{x}))\, V(\boldsymbol{x}) + L &= (1 - h(\boldsymbol{x}))^{\frac{1}{2}}\, V(\boldsymbol{x}) \cdot (1 - h(\boldsymbol{x}))^{\frac{1}{2}} (1 + L/[(1 - h(\boldsymbol{x}))\, V(\boldsymbol{x})]) \\
&\leq \left(1 - \tfrac{1}{2} h(\boldsymbol{x})\right) V(\boldsymbol{x}) \cdot (1 - h(\boldsymbol{x}))^{\frac{1}{2}} (1 + L/[(1 - h(\boldsymbol{x}))\, V(\boldsymbol{x})]) \\
&\leq \left(1 - \tfrac{1}{2} h(\boldsymbol{x})\right) V(\boldsymbol{x})
\end{aligned}$$

for all $|\boldsymbol{x}|$ large enough, where the first inequality is based on the inequality $(1 - x)^a \leq 1 - ax$ (which holds for $a, x \in [0, 1]$) and the second inequality is justified by showing that the inequality

$$H(\boldsymbol{x}) \overset{def}{=} (1 - h(\boldsymbol{x}))^{\frac{1}{2}} (1 + L/[(1 - h(\boldsymbol{x}))\, V(\boldsymbol{x})]) < 1$$

holds for all $|\boldsymbol{x}|$ large enough. To show this, note first that

$$H(\boldsymbol{x}) = (1 - h(\boldsymbol{x}))^{\frac{1}{2}} + L/[(1 - h(\boldsymbol{x}))^{1/2}\, V(\boldsymbol{x})] \leq 1 - \tfrac{1}{2} h(\boldsymbol{x}) + L/[(1 - h(\boldsymbol{x}))^{1/2}\, V(\boldsymbol{x})],$$

so that it suffices to show that, for all $|\boldsymbol{x}|$ large enough, the right hand side of the last inequality is smaller than one or, equivalently, that $L < \frac{1}{2} h(\boldsymbol{x}) (1 - h(\boldsymbol{x}))^{\frac{1}{2}} V(\boldsymbol{x})$. This holds for all $|\boldsymbol{x}|$ large enough due to the definitions of $V(\boldsymbol{x})$ and $h(\boldsymbol{x})$ which imply that, as $|\boldsymbol{x}| \to \infty$, $V(\boldsymbol{x}) \to \infty$ at an exponential rate (see (30)) whereas $h(\boldsymbol{x}) \to 0$ at a logarithmic rate (see the above definition of $h(\boldsymbol{x})$).

We can therefore write inequality (37), for all $|\boldsymbol{x}|$ large enough, as

$$E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq \left(1 - \tfrac{1}{2} h(\boldsymbol{x})\right) V(\boldsymbol{x}).$$

As the right hand side is bounded when $\boldsymbol{x}$ belongs to any compact set, this further implies that there exist positive constants M and b such that for $C = \{\boldsymbol{x} \in \mathbb{R}^p : |\boldsymbol{x}| \leq M\}$ and for all $\boldsymbol{x} \in \mathbb{R}^p$

$$E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq V(\boldsymbol{x}) - \phi_1 (V(\boldsymbol{x})) + b \mathbf{1}_C(\boldsymbol{x}), \tag{38}$$

where

$$\phi_1(V(\boldsymbol{x})) = \tfrac{1}{2}h(\boldsymbol{x})V(\boldsymbol{x}) = \tfrac{1}{2}c_\phi(1+\ln V(\boldsymbol{x}))^{-\alpha} V(\boldsymbol{x}). \tag{39}$$

Now note that we can always find positive constants v_0 and c such that the function $\phi(v) = c(v+v_0)(\ln(v+v_0))^{-\alpha}$ is a concave increasing differentiable function for all $v \geq 1$ and such that

$$\phi_1(v) = \tfrac{1}{2}c_\phi v(1+\ln(v))^{-\alpha} \geq c(v+v_0)(\ln(v+v_0))^{-\alpha} = \phi(v)$$

for large enough v. Therefore, potentially redefining M, b, and C,

$$E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq V(\boldsymbol{x}) - \phi(V(\boldsymbol{x})) + b\mathbf{1}_C(\boldsymbol{x}).$$

Thus, we have verified Condition D (with $\alpha = \rho/b_3 - 1$). The result follows from Theorem 1.

Part (ii). Now $\rho = \kappa_0$ and, as in the proof of Theorem 3.3(ii) in Douc et al. (2004, p. 1373), many results in the proof of case $\rho > \kappa_0$ can be used. Again, we choose $b_3 = \kappa_0 \wedge (2-\rho)$, noting that now $b_3 = \kappa_0$ and that the notation κ_0 will be used below instead of b_3. Also, the function $V(\boldsymbol{x}) = V_1(z_1) + V_2(\boldsymbol{z}_2)$ is as in the case $\rho > \kappa_0$, and we need to bound the two terms in (31).

Step 1: Bounding $E[V_1(g(z_1)+\varepsilon_1)]$ in (31). Exactly as in Part (i), Step 1, it again holds that, for $|z_1| > M_0$,

$$\begin{aligned} V_1(g(z_1)) - V_1(z_1) &\leq \left(-b_1 r\kappa_0 |z_1|^{\kappa_0-\rho} + \tfrac{1}{2}b_1^2 r^2 \kappa_0^2 |z_1|^{2(\kappa_0-\rho)}\right)V_1(z_1) \\ &= \left(-b_1 r\kappa_0 + \tfrac{1}{2}b_1^2 r^2\kappa_0^2\right)V_1(z_1) \end{aligned}$$

and, for large $|z_1|$,

$$E[V_1(g(z_1)+\varepsilon_1)] - V_1(g(z_1)) \leq b_1^2\kappa_0^2 E\left[\varepsilon_1^2 V_1(\varepsilon_1)\right]|z_1|^{2\kappa_0-2} V_1(z_1).$$

Hence, for large $|z_1|$,

$$E[V_1(g(z_1)+\varepsilon_1)] \leq V_1(z_1) + k(z_1)V_1(z_1),$$

where now

$$k(z_1) = -b_1 r\kappa_0 + \tfrac{1}{2}b_1^2 r^2\kappa_0^2 + b_1^2\kappa_0^2 E\left[\varepsilon_1^2 V_1(\varepsilon_1)\right]|z_1|^{2\kappa_0-2}.$$

Due to Assumption 2(a) and the choice of b_3, the condition $E\left[|\varepsilon_1|^2 V_1(\varepsilon_1)\right] < \infty$ can be achieved by choosing the value of b_1 small enough or, specifically, assuming $b_1 = \tilde{b}_1 < \beta_0$. Furthermore, as $\kappa_0 \in (0,1]$, by choosing the value of b_1 small enough the function $k(z_1) \in (-1,0)$ and is bounded away from -1 and 0 for any $|z_1|$ large enough. Therefore, for some $\delta_1 \in (0,1)$,

$$E[V_1(g(z_1)+\varepsilon_1)] \leq V_1(z_1) - \delta_1 V_1(z_1)$$

for all sufficiently large $|z_1|$.

Step 2: Bounding $V_\epsilon(\boldsymbol{x})E[V_1(g(z_1)+\varepsilon_1)]$ in (31). For the first term on the right hand side of (31) we obtain, for $|z_1|$ large,

$$V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1)+\varepsilon_1\right)\right] \le \exp\left\{b_1\left|\epsilon(\boldsymbol{x})\boldsymbol{x}\right|^{\kappa_0}\right\}(1-\delta_1)V_1(z_1) = \tfrac{1}{2}(1-\delta_1)\exp\left\{b_1\left|\epsilon(\boldsymbol{x})\boldsymbol{x}\right|^{\kappa_0}+b_1\left|z_1\right|^{\kappa_0}\right\}.$$

Write $1-\delta_1=(1-\delta_1)^{1/2}(1-\delta_1)^{1/2}\le(1-\frac{1}{2}\delta_1)^2$ and note that $1-\frac{1}{2}\delta_1=\exp\left\{\ln\left(1-\frac{1}{2}\delta_1\right)\right\}\le\exp\left\{-\frac{1}{2}\delta_1\right\}$ to obtain

$$V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1)+\varepsilon_1\right)\right] \le \tfrac{1}{2}\left(1-\tfrac{1}{2}\delta_1\right)\exp\left\{-\tfrac{1}{2}\delta_1+b_1\left|z_1\right|^{\kappa_0}+b_1\left|\epsilon(\boldsymbol{x})\boldsymbol{x}\right|^{\kappa_0}\right\}.$$

As Assumption 1(ii.a) now holds with $d=1$, we have $|\epsilon(\boldsymbol{x})|=o(|\boldsymbol{x}|^{-1})$ and

$$b_1\left|\epsilon(\boldsymbol{x})\boldsymbol{x}\right|^{\kappa_0}\le o(|\boldsymbol{x}|^{-\kappa_0})b_1\left|z_1\right|^{\kappa_0}+o(|\boldsymbol{x}|^{-\kappa_0})b_1\left\|\boldsymbol{z}_2\right\|_*^{\kappa_0}$$

(cf. the similar inequality in the proof of case $\rho>\kappa_0$, Step 2). Therefore, for $|z_1|$ large,

$$-\tfrac{1}{2}\delta_1+b_1\left|z_1\right|^{\kappa_0}+b_1\left|\epsilon(\boldsymbol{x})\boldsymbol{x}\right|^{\kappa_0}\le\left(1-\delta_1(z_1)\right)b_1\left|z_1\right|^{\kappa_0}+o(|\boldsymbol{x}|^{-\kappa_0})b_1\left\|\boldsymbol{z}_2\right\|_*^{\kappa_0},$$

where

$$\delta_1(z_1)=\frac{\delta_1}{2b_1\left|z_1\right|^{\kappa_0}}+o(|\boldsymbol{x}|^{-\kappa_0})=\frac{\delta_1+o(1)}{2b_1\left|z_1\right|^{\kappa_0}}$$

with $\delta_1(z_1)\in(0,1)$ and $\delta_1(z_1)^{-1}o(|\boldsymbol{x}|^{-\kappa_0})=o(1)$ holding (here, as well as above, the term $o(1)$ is obtained because $|z_1|^{\kappa_0}o(|\boldsymbol{x}|^{-\kappa_0})=o(1)$ by arguments similar to those used in the case $\rho>\kappa_0$, Step 2).

Thus, we can conclude that, for $|z_1|$ large,

$$\begin{aligned}
&V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1)+\varepsilon_1\right)\right]\\
&\quad\le\tfrac{1}{2}\left(1-\tfrac{1}{2}\delta_1\right)\exp\left\{\left(1-\delta_1(z_1)\right)b_1\left|z_1\right|^{\kappa_0}+\delta_1(z_1)\delta_1(z_1)^{-1}o(|\boldsymbol{x}|^{-\kappa_0})b_1\left\|\boldsymbol{z}_2\right\|_*^{\kappa_0}\right\}\\
&\quad\le\tfrac{1}{2}\left(1-\tfrac{1}{2}\delta_1\right)\left(1-\delta_1(z_1)\right)\exp\left\{b_1\left|z_1\right|^{\kappa_0}\right\}+\tfrac{1}{2}\left(1-\tfrac{1}{2}\delta_1\right)\delta_1(z_1)\exp\left\{\delta_1(z_1)^{-1}o(|\boldsymbol{x}|^{-\kappa_0})b_1\left\|\boldsymbol{z}_2\right\|_*^{\kappa_0}\right\}\\
&\quad\le\tfrac{1}{2}\left(1-\tfrac{1}{2}\delta_1\right)\exp\left\{b_1\left|z_1\right|^{\kappa_0}\right\}+\tfrac{1}{2}\exp\left\{o(1)b_1\left\|\boldsymbol{z}_2\right\|_*^{\kappa_0}\right\},
\end{aligned}$$

where the second inequality is due to the convexity of the exponential function. To state this more formally, we can find $M_1\ge M_0$ and some (finite) $\overline{M}_1$, such that

$$V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1)+\varepsilon_1\right)\right]\le V_1(z_1)-\tfrac{1}{2}\delta_1V_1(z_1)+\tfrac{1}{2}\exp\left\{o(1)b_1\left\|\boldsymbol{z}_2\right\|_*^{\kappa_0}\right\}+\overline{M}_1\mathbf{1}_{C_1}(z_1),$$

where $\delta_1\in(0,1)$ and $C_1=\{z_1\in\mathbb{R}:|z_1|\le M_1\}$ (cf. the proof of part (ii) in Douc et al. (2004, p. 1373)). Moreover, as in case $\rho>\kappa_0$ (the beginning of Step 4), the term $o(1)$ on the right hand side converges to zero as $|\boldsymbol{z}_2|\to\infty$.

Step 3: Bounding $V_2\left(\boldsymbol{\Pi}_1\boldsymbol{z}_2+z_1\boldsymbol{\iota}_{p-1}\right)$ in (31). As in the the proof of case $\rho>\kappa_0$, Step 3, assume that the value of b_1 is fixed to $\tilde{b}_1$ specified above. Repeating the arguments in the proof of case $\rho>\kappa_0$, Step 3, we first obtain

$$V_2\big(\boldsymbol{\Pi}\boldsymbol{z}_2+z_1\boldsymbol{\iota}_{p-1}\big)\le\tfrac{1}{2}\exp\left\{b_{2,1}\left|z_1\right|^{\kappa_0}\right\}+\tfrac{1}{2}\exp\left\{b_{2,2}\eta^{\kappa_0}\left\|\boldsymbol{z}_2\right\|_*^{\kappa_0}\right\}\overset{def}{=}V_{2,1}(z_1)+V_{2,2}(\boldsymbol{z}_2),$$

where $b_{2,1}=b_2\left\|\boldsymbol{\iota}_{p-1}\right\|_*^{\kappa_0}/\tau_1$ and $b_{2,2}=b_2/\tau_2$ with $\tau_1\in(0,1)$ and $\tau_2=1-\tau_1$. Also, as in case

$\rho > \kappa_0$, we can choose $\tau_2 \in (\eta^{\kappa_0}, 1)$ so that $b_{2,2}\eta^{\kappa_0} = b_2\eta^{\kappa_0}/\tau_2 < b_2$, and the value of b_2 so small that $b_{2,1} = b_2 \|\boldsymbol{\iota}_{p-1}\|_*^{\kappa_0}/\tau_1 < b_1 = \tilde{b}_1$ with $\tilde{b}_1$ as fixed above.

We next bound $V_{2,1}(z_1)$ and $V_{2,2}(\boldsymbol{z}_2)$. Arguments similar to those used in the corresponding proof of case $\rho > \kappa_0$, Step 3, apply but the bound obtained for $V_{2,1}(z_1)$ simplifies. Specifically,

$$V_{2,1}(z_1) \leq \epsilon V_1(z_1) \quad \text{and} \quad V_{2,2}(\boldsymbol{z}_2) \leq \eta_2 V_2(\boldsymbol{z}_2),$$

where the first inequality holds for any $\epsilon > 0$ as long as $|z_1|$ is large enough and the second inequality holds for some $\eta_2 \in (0,1)$ and $\|\boldsymbol{z}_2\|_*$ bounded away from zero. These inequalities can be written as

$$V_{2,1}(z_1) \leq \epsilon V_1(z_1) + \overline{M}_1 \mathbf{1}_{C_1}(z_1) \quad \text{and} \quad V_{2,2}(\boldsymbol{z}_2) \leq V_2(\boldsymbol{z}_2) - (1-\eta_2)V_2(\boldsymbol{z}_2) + \overline{M}_2 \mathbf{1}_{C_2}(\boldsymbol{z}_2),$$

where, for simplicity, we have assumed that the term $\overline{M}_1 \mathbf{1}_{C_1}(z_1)$ can be the same as at the end of Step 2 and where $C_2 = \{\boldsymbol{z}_2 \in \mathbb{R}^{p-1} : |\boldsymbol{z}_2| \leq M_2\}$ with M_2 and $\overline{M}_2$ some positive and finite constants. Thus, we can conclude that

$$V_2\big(\boldsymbol{\Pi}\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\big) \leq \epsilon V_1(z_1) + V_2(\boldsymbol{z}_2) - (1-\eta_2)V_2(\boldsymbol{z}_2) + \overline{M}_1 \mathbf{1}_{C_1}(z_1) + \overline{M}_2 \mathbf{1}_{C_2}(\boldsymbol{z}_2).$$

Step 4: Bounding $E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right]$ in (31). The bounds obtained for $V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right]$ and $V_2\big(\boldsymbol{\Pi}\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\big)$ in Steps 2 and 3, respectively, yield

$$\begin{aligned} E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] &= V_\epsilon(\boldsymbol{x})E\left[V_1\left(g(z_1) + \varepsilon_1\right)\right] + V_2\big(\boldsymbol{\Pi}\boldsymbol{z}_2 + z_1\boldsymbol{\iota}_{p-1}\big) \\ &\leq V_1(z_1) - \tfrac{1}{2}\delta_1 V_1(z_1) + \tfrac{1}{2}\exp\left\{o(1)b_1\|\boldsymbol{z}_2\|_*^{\kappa_0}\right\} + \epsilon V_1(z_1) \\ &\quad + V_2(\boldsymbol{z}_2) - (1-\eta_2)V_2(\boldsymbol{z}_2) + 2\overline{M}_1 \mathbf{1}_{C_1}(z_1) + \overline{M}_2 \mathbf{1}_{C_2}(\boldsymbol{z}_2). \end{aligned}$$

As the value of $\epsilon > 0$ can be made as close to zero as desired (by only choosing $|z_1|$ large enough and independently of choices made for any other parameters), we can assume that $\epsilon < \frac{1}{2}\delta_1$ so that

$$-\tfrac{1}{2}\delta_1 V_1(z_1) + \epsilon V_1(z_1) \leq -\delta_2 V_1(z_1)$$

holds with some $\delta_2 \in (0,1)$. Moreover, as in the proof of case $\rho > \kappa_0$, Step 4,

$$\tfrac{1}{2}\exp\{o(1)\|\boldsymbol{z}_2\|_*^{b_3}\} - (1-\eta_2)V_2(\boldsymbol{z}_2) \leq -\eta_3 V_2(\boldsymbol{z}_2),$$

$\eta_3 \in (0,1)$. Thus, defining $\tilde{\lambda} = \delta_2 \wedge \eta_3 \in (0,1)$ and $L \geq 2\overline{M}_1 + \overline{M}_2$ we find that

$$\begin{aligned} E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] &\leq V_1(z_1(\boldsymbol{x})) - \delta_2 V_1(z_1(\boldsymbol{x})) + V_2(\boldsymbol{z}_2(\boldsymbol{x})) - \eta_3 V_2(\boldsymbol{z}_2(\boldsymbol{x})) + L, \\ &\leq V(\boldsymbol{x}) - \tilde{\lambda}V(\boldsymbol{x}) + L. \end{aligned}$$

We can write the above inequality as

$$E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq (1-\tilde{\lambda})^{\frac{1}{2}}V(\boldsymbol{x}) \cdot (1-\tilde{\lambda})^{\frac{1}{2}}\big(1 + L/[(1-\tilde{\lambda})V(\boldsymbol{x})]\big),$$

from which it follows that, for all $|\boldsymbol{x}|$ large enough, $E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq (1-\tilde{\lambda})^{\frac{1}{2}}V(\boldsymbol{x})$, implying that there exist positive constants M and b such that, for $C = \{\boldsymbol{x} \in \mathbb{R}^p : |\boldsymbol{x}| \leq M\}$,

$$E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] \leq (1-\tilde{\lambda})^{\frac{1}{2}} V(\boldsymbol{x}) + b\mathbf{1}_C(\boldsymbol{x}).$$

Defining $\lambda = 1 - (1-\tilde{\lambda})^{\frac{1}{2}} \in (0,1)$ we can conclude that Condition D holds with $\phi(v) = \lambda v$ and therefore Theorem 1(i) shows that the Markov chain $\boldsymbol{y}_t$ is geometrically ergodic and the convergence (8) holds with $f(\boldsymbol{x}) = V(\boldsymbol{x}) = V_1(z_1(\boldsymbol{x})) + V_2(\boldsymbol{z}_2(\boldsymbol{x}))$.

Case $p = 1$: When $p = 1$ we have $\boldsymbol{x} = x_1 = u$ and we simply write x for any of these. In this case, model (16) reduces to $y_t = y_{t-1} + \tilde{g}(y_{t-1}) + \varepsilon_t$, Assumption 1(i) becomes redundant, Assumption 1(ii.a) is automatically satisfied with $g(x) = x + \tilde{g}(x)$, $\epsilon(x) = 0$, and d redundant (as long as the condition $|g(x)| \to \infty$ as $|x| \to \infty$ still holds), and Assumptions 1(ii.b) and 2 are as in the case $p \geq 2$. In other words, the model can be written as $y_t = g(y_{t-1}) + \varepsilon_t$ with g satisfying Assumption 1(ii.b) as well as $|g(x)| \to \infty$ as $|x| \to \infty$. This also means that the assumptions of Theorem 3.3 in Douc et al. (2004) are satisfied except for the continuity of g required in their Assumption 3.4. However, in our case this assumption is not needed because the boundedness of g on compact subsets of $\mathbb{R}$ implied by our Assumption 1(ii) actually suffices.

First consider the case $\rho > \kappa_0$. Proceeding as in the proof of Theorem 3.3(i) of Douc et al. (2004) we can conclude that there exist positive constants M and b such that, for $C = \{x \in \mathbb{R} : |x| \leq M\}$ and for all $x \in \mathbb{R}$,

$$E[V(y_1) \mid y_0 = x] \leq V(x) - \phi_1(V(x)) + b\mathbf{1}_C(x), \tag{40}$$

where $\phi_1(V(x)) = \tilde{c}_\phi (1 + \ln V(x))^{-\alpha} V(x)$ with $\alpha = \rho/b_3 - 1 > 0$ and some $\tilde{c}_\phi > 0$ (see the top of p. 1373 of Douc et al. (2004) and note also our additional assumption $|g(x)| \to \infty$ as $|x| \to \infty$). Comparing this with (38) and (39) at the end of the proof of part (i) shows that we can continue as therein and conclude that Condition D is satisfied with $V(x) = \exp\{b_1 |x|^{b_3}\}$ and $\phi(v) = c(v + v_0)(\ln(v + v_0))^{-\alpha}$ (for some $c, v_0 > 0$ and $\alpha = \rho/b_3 - 1 > 0$). The result of part (i) now follows from Theorem 1(ii).

Now consider the case $\rho = \kappa_0$. As in the proof of Theorem 3.3(ii) of Douc et al. (2004) we can conclude that (40) holds with $\phi_1(V(x)) = \lambda V(x)$ and some $\lambda > 0$, and with M, b, and C redefined (see the middle of p. 1373 of Douc et al. (2004) and note again the above-mentioned additional assumption). The result of part (ii) now follows from Theorem 1(i). ■

Proof of Theorem 3. First note that our Assumption 2(b) implies Assumptions (NSS 1) and (NSS 4) of Fort and Moulines (2003). Also, in the same way as in the proof of Theorem 2 we can show that the Markov chain $\boldsymbol{y}_t$ is a ψ-irreducible and aperiodic T-chain with ψ the Lebesgue measure, and that all compact sets of $\mathcal{B}(\mathbb{R}^p)$ are petite. This, in turn, implies that Assumption (NSS 2) of Fort and Moulines (2003) holds. These facts together with Assumption 1 are used below to verify Assumption (NSS 3) of Fort and Moulines (2003) which enables us to apply Lemma 3 of that paper.

As $V(\boldsymbol{x}) = 1 + |z_1|^{s_0} + s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0}$ we have (cf. the beginning of the proof of Theorem 2)

$$E[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}] = 1 + E[|\overline{g}(\boldsymbol{x}) + \varepsilon_1|^{s_0}] + s_1 \|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0}. \tag{41}$$

In this case it appears convenient to start with bounding the latter term on the right hand side.

Step 1: Bounding $s_1 \|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0}$ in (41). First note that $\alpha = 1 - \rho/s_0 \in (0,1)$ because $0 < \rho < s_0$ is assumed. We consider separately the cases where $\alpha s_0 \le 1$ and $\alpha s_0 > 1$, and show that there exist constants $\eta_0 \in (0,1)$ and $\bar{s}_1 > 0$ such that

$$s_1 \|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0} \le s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - \eta_0 s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} + \bar{s}_1 |z_1|^{\alpha s_0} \tag{42}$$

holds for both $\alpha s_0 \le 1$ and $\alpha s_0 > 1$. Moreover, the value of $\bar{s}_1$ can be chosen as close to zero as desired.

First consider the case $\alpha s_0 \le 1$ and assume that $s_1 < 1$. Denoting $\tilde{s}_1 = s_1 \|\boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0}$ we obtain (cf. the proof of Theorem 2, the beginning of Step 3)

$$s_1 \|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0} \le s_1 \eta^{\alpha s_0} \|\boldsymbol{z}_2\|_*^{\alpha s_0} + \tilde{s}_1 |z_1|^{\alpha s_0} = s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - \eta_1 s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} + \tilde{s}_1 |z_1|^{\alpha s_0},$$

where $\eta \in (0,1)$ by assumption and $\eta_1 = 1 - \eta^{\alpha s_0} \in (0,1)$ which shows that inequality (42) holds with $\eta_0 = \eta_1$ and $\bar{s}_1 = \tilde{s}_1$. Also, the value of $\tilde{s}_1$ can be made as close to zero as desired by choosing s_1 small enough.

Now consider the case $\alpha s_0 > 1$. Here $s_1 < 1$ is still assumed and $s_0 > 1$ must hold because $\alpha \in (0,1)$. Write

$$\begin{aligned} s_1 \|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0} &= s_1 \left(\|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha}\right)^{s_0} \\ &\le s_1 \left(\|\boldsymbol{\Pi}_1 \boldsymbol{z}_2\|_*^{\alpha} + \|\boldsymbol{\iota}_{p-1}\|_*^{\alpha} |z_1|^{\alpha}\right)^{s_0} \\ &\le \left(s_1^{1/s_0} \eta^{\alpha} \|\boldsymbol{z}_2\|_*^{\alpha} + s_1^{1/s_0} \|\boldsymbol{\iota}_{p-1}\|_*^{\alpha} |z_1|^{\alpha}\right)^{s_0}, \end{aligned}$$

where $\eta \in (0,1)$ again holds by assumption. Let $\tau_1 \in (0,1)$ and $\tau_2 = 1 - \tau_1$, and denote $s_{1,1} = s_1^{1/s_0} \|\boldsymbol{\iota}_{p-1}\|_*^{\alpha} / \tau_1$ and $s_{1,2} = s_1^{1/s_0}/\tau_2$. Then,

$$\begin{aligned} s_1 \|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0} &\le (\tau_1 s_{1,1} |z_1|^{\alpha} + \tau_2 s_{1,2} \eta^{\alpha} \|\boldsymbol{z}_2\|_*^{\alpha})^{s_0} \\ &\le \tau_1 s_{1,1}^{s_0} |z_1|^{\alpha s_0} + \tau_2 s_{1,2}^{s_0} \eta^{\alpha s_0} \|\boldsymbol{z}_2\|_*^{\alpha s_0} \\ &\le s_{1,1}^{s_0} |z_1|^{\alpha s_0} + s_{1,2}^{s_0} \eta^{\alpha s_0} \|\boldsymbol{z}_2\|_*^{\alpha s_0}, \end{aligned}$$

where the second inequality is justified by the convexity of the function $|x| \mapsto |x|^{s_0}$ for $s_0 > 1$.

Next, as $\eta^{\alpha} < 1$, we can choose $\tau_2 \in (\eta^{\alpha}, 1)$ so that $s_{1,2}^{s_0} \eta^{\alpha s_0} = s_1 \eta^{\alpha s_0} / \tau_2^{s_0} < s_1$. Denoting $\eta_2 = 1 - (\eta^{\alpha}/\tau_2)^{s_0}$ we have $\eta_2 \in (0,1)$ and

$$s_{1,2}^{s_0} \eta^{\alpha s_0} \|\boldsymbol{z}_2\|_*^{\alpha s_0} = s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - s_1 (1 - \eta^{\alpha s_0}/\tau_2^{s_0}) \|\boldsymbol{z}_2\|_*^{\alpha s_0} = s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - \eta_2 s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0},$$

and we can conclude that

$$s_1 \|\boldsymbol{\Pi}_1 \boldsymbol{z}_2 + z_1 \boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0} \le s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - \eta_2 s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} + s_{1,1}^{s_0} |z_1|^{\alpha s_0}.$$

Thus, inequality (42) holds with $\eta_0 = \eta_2$ and $\bar{s}_1 = s_{1,1}^{s_0}$. Above we fixed the value of τ_2, and hence also the value of τ_1, but we are still free to choose the value of s_1 and make $s_{1,1}^{s_0} = s_1 \|\boldsymbol{\iota}_{p-1}\|_*^{\alpha s_0} / \tau_1^{s_0} < 1$ as close to zero as desired by choosing s_1 small enough. From now on, we assume that $\eta_0 = \eta_1 \wedge \eta_2$ and $\bar{s}_1 = \tilde{s}_1 \vee s_{1,1}^{s_0}$ so that inequality (42) applies irrespective of whether $\alpha s_0 \leq 1$ or $\alpha s_0 > 1$, and the value of $\bar{s}_1$ can be chosen arbitrarily close to zero.

Step 2: Bounding $E\left[|\bar{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right]$ in (41). Consider the cases $s_0 < 1$ and $s_0 \geq 1$ separately. When $s_0 < 1$, the definition of the function $\bar{g}$, the triangle inequality, and Assumption 1(ii.a) yield

$$\begin{aligned} E\left[|\bar{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right] &= E\left[|\bar{g}(\boldsymbol{x})-g(z_1)+g(z_1)+\varepsilon_1|^{s_0}\right] \\ &\leq E\left[|\bar{g}(\boldsymbol{x})-g(z_1)|^{s_0}+|g(z_1)+\varepsilon_1|^{s_0}\right] \\ &\leq |\epsilon(\boldsymbol{x})\boldsymbol{x}|^{s_0}+E\left[|g(z_1)+\varepsilon_1|^{s_0}\right]. \end{aligned} \tag{43}$$

When $s_0 \geq 1$, we can use Minkowski's inequality and obtain

$$\begin{aligned} \left(E\left[|\bar{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right]\right)^{1/s_0} &= \left(E\left[|\bar{g}(\boldsymbol{x})-g(z_1)+g(z_1)+\varepsilon_1|^{s_0}\right]\right)^{1/s_0} \\ &\leq |\bar{g}(\boldsymbol{x})-g(z_1)|+\left(E\left[|g(z_1)+\varepsilon_1|^{s_0}\right]\right)^{1/s_0} \\ &\leq |\epsilon(\boldsymbol{x})\boldsymbol{x}|+\left(E\left[|g(z_1)+\varepsilon_1|^{s_0}\right]\right)^{1/s_0}. \end{aligned} \tag{44}$$

The next step is to bound the expectation $E\left[|g(z_1)+\varepsilon_1|^{s_0}\right]$. Assumption 1(ii.b) ensures that the function g satisfies the conditions in Assumption (NSS 3) of Fort and Moulines (2003) which (together with other assumptions of the theorem) implies that we can use Lemma 3 of that paper. Thus, as $\alpha s_0 = s_0 - \rho$, inequality (36) in that lemma shows that

$$E\left[|g(z_1)+\varepsilon_1|^{s_0}\right] \leq |z_1|^{s_0} - \lambda |z_1|^{\alpha s_0}\left(1+\tilde{\epsilon}(z_1)\right),$$

where $\tilde{\epsilon}(z_1) \to 0$ as $|z_1| \to \infty$ and $\lambda > 0$ (to see this, note that the cases (i)–(iii) in our Theorem 3 correspond to the cases (i)–(iii) in Lemma 3 of Fort and Moulines (2003) so that the result is obtained with $\lambda = s_0 r$ in cases (i) and (ii) and with $\lambda = s_0 r - \frac{1}{2}s_0(s_0-1)E[\varepsilon_1^2]$, which is positive by assumption, in case (iii)). Thus, the above inequality implies that, for $|z_1|$ large,

$$E\left[|g(z_1)+\varepsilon_1|^{s_0}\right] \leq |z_1|^{s_0} - \tilde{\lambda}|z_1|^{\alpha s_0}, \tag{45}$$

where $\tilde{\lambda} > 0$ and, without loss of generality, we can assume that $\tilde{\lambda} \leq 1$ also holds. Note that this inequality holds for both $s_0 < 1$ and $s_0 \geq 1$; these two cases will be treated separately below.

Case $s_0 < 1$. First recall from the proof of Theorem 2, Step 2, that

$$|\epsilon(\boldsymbol{x})\boldsymbol{x}| \leq |\epsilon_1(\boldsymbol{x})|\,|z_1| + |\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*,$$

where $\epsilon_1(\boldsymbol{x}) = c_*\epsilon(\boldsymbol{x})$, $c_* > 0$. Using (43), (45), and the assumption $s_0 < 1$, we find that, for $|z_1|$ large,

$$\begin{aligned} E\left[|\bar{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right] &\leq |z_1|^{s_0} - \tilde{\lambda}|z_1|^{\alpha s_0} + |\epsilon(\boldsymbol{x})\boldsymbol{x}|^{s_0} \\ &\leq |z_1|^{s_0} - \tilde{\lambda}|z_1|^{\alpha s_0} + |\epsilon_1(\boldsymbol{x})|^{s_0}|z_1|^{s_0} + |\epsilon_1(\boldsymbol{x})|^{s_0}\|\boldsymbol{z}_2\|_*^{s_0} \\ &\leq |z_1|^{s_0} - \tilde{\lambda}|z_1|^{\alpha s_0} + |\epsilon_1(\boldsymbol{x})|^{s_0}|z_1|^{\rho}|z_1|^{\alpha s_0} + |\epsilon_1(\boldsymbol{x})|^{s_0}\|\boldsymbol{z}_2\|_*^{\rho}\|\boldsymbol{z}_2\|_*^{\alpha s_0}, \end{aligned}$$

where the last inequality follows because $\alpha = 1 - \rho/s_0$ so that $\alpha s_0 = s_0 - \rho$. As $|\epsilon_1(\boldsymbol{x})| =$

$o(|\boldsymbol{x}|^{-\rho/s_0})$ by assumption, we have, for $|z_1|$ large, $|\epsilon_1(\boldsymbol{x})|^{s_0}|z_1|^\rho = o(|\boldsymbol{x}|^{-\rho})|z_1|^\rho < \tilde{\lambda}$, and thus

$$E\left[|\overline{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right] \le |z_1|^{s_0} - \tilde{\lambda}_1 |z_1|^{\alpha s_0} + o(1)\,\|\boldsymbol{z}_2\|_*^{\alpha s_0}, \tag{46}$$

where $\tilde{\lambda}_1 \in (0,1)$ and $o(1) \to 0$ as $|\boldsymbol{x}| \to \infty$ (the upper bound of $\tilde{\lambda}_1$ follows because $\tilde{\lambda} \le 1$ was assumed above and the term $o(1)$ is obtained as in the proof of Theorem 2, Step 2).

Case $s_0 \ge 1$. When $s_0 \ge 1$, inequalities (44) and (45) imply that, for $|z_1|$ large,

$$\begin{aligned}
\left(E\left[|\overline{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right]\right)^{1/s_0} &\le \left(E\left[|g(z_1)+\varepsilon_1|^{s_0}\right]\right)^{1/s_0} + |\epsilon(\boldsymbol{x})\boldsymbol{x}| \\
&\le (|z_1|^{s_0} - \tilde{\lambda}|z_1|^{\alpha s_0})^{1/s_0} + |\epsilon_1(\boldsymbol{x})|\,|z_1| + |\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_* \\
&= |z_1|\,(1-\tilde{\lambda}|z_1|^{-\rho})^{1/s_0} + |\epsilon_1(\boldsymbol{x})|\,|z_1| + |\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_* \\
&\le |z_1|\left(1-\frac{\tilde{\lambda}}{s_0}|z_1|^{-\rho}\right) + |\epsilon_1(\boldsymbol{x})|\,|z_1| + |\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*\,.
\end{aligned}$$

Here the equality is again due to the definition of α which implies $\alpha s_0 = s_0 - \rho$, and the last inequality follows because $(1-u)^a \le 1 - au$ holds for all $0 \le u, a \le 1$. As $|\epsilon_1(\boldsymbol{x})| = o(|\boldsymbol{x}|^{-\rho})$ by assumption, we have, for $|z_1|$ large enough, $|\epsilon_1(\boldsymbol{x})|\,|z_1| = |\epsilon_1(\boldsymbol{x})|\,|z_1|^\rho\,|z_1|^{1-\rho} < \frac{\tilde{\lambda}}{s_0}|z_1|^{1-\rho}$, and

$$\left(E\left[|\overline{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right]\right)^{1/s_0} \le |z_1|\left(1-\tilde{\lambda}_2|z_1|^{-\rho}\right) + |\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*\,, \tag{47}$$

where $\tilde{\lambda}_2 \in (0,1)$. As the term $1-\tilde{\lambda}_2|z_1|^{-\rho}$ in (47) is positive, we can write

$$1-\tilde{\lambda}_2|z_1|^{-\rho} = \left(1-\tilde{\lambda}_2|z_1|^{-\rho}\right)^{1/2}\left(1-\tilde{\lambda}_2|z_1|^{-\rho}\right)^{1/2} \le \left(1-\tfrac{1}{2}\tilde{\lambda}_2|z_1|^{-\rho}\right)^2,$$

and arguments similar to those in the proof of Theorem 2, Step 2, can be used. Thus, we define $\tau_1(z_1) = 1-\frac{1}{2}\tilde{\lambda}_2|z_1|^{-\rho}$ and $\tau_2(z_1) = 1-\tau_1(z_1)$, and express inequality (47) as

$$\left(E\left[|\overline{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right]\right)^{1/s_0} \le \tau_1(z_1)\,|z_1|\left(1-\tfrac{1}{2}\tilde{\lambda}_2|z_1|^{-\rho}\right) + \tau_2(z_1)\tau_2(z_1)^{-1}|\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*\,.$$

From this we can conclude that, for $|z_1|$ large,

$$\begin{aligned}
E\left[|\overline{g}(\boldsymbol{x})+\varepsilon_1|^{s_0}\right] &\le \left[\tau_1(z_1)\,|z_1|\left(1-\tfrac{1}{2}\tilde{\lambda}_2|z_1|^{-\rho}\right) + \tau_2(z_1)\tau_2(z_1)^{-1}|\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*\right]^{s_0} \\
&\le \tau_1(z_1)\,|z_1|^{s_0}\left(1-\tfrac{1}{2}\tilde{\lambda}_2|z_1|^{-\rho}\right)^{s_0} + \tau_2(z_1)\left(\tau_2(z_1)^{-1}|\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*\right)^{s_0} \\
&\le |z_1|^{s_0}\left(1-\tfrac{1}{2}\tilde{\lambda}_2|z_1|^{-\rho}\right)^{s_0} + \tau_2(z_1)\left(\tau_2(z_1)^{-1}|\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*\right)^{s_0} \\
&\le |z_1|^{s_0}\left(1-\tfrac{1}{2}\tilde{\lambda}_2|z_1|^{-\rho}\right) + \tau_2(z_1)\left(\tau_2(z_1)^{-1}|\epsilon_1(\boldsymbol{x})|\,\|\boldsymbol{z}_2\|_*\right)^{s_0}.
\end{aligned}$$

Here the second inequality is due to the convexity of the function $|x| \mapsto |x|^{s_0}$, $s_0 \ge 1$, and the last one follows because $s_0 \ge 1$. By the definition of $\tau_2(z_1)$, $\tau_2(z_1)^{-1}|\epsilon_1(\boldsymbol{x})| = (2/\tilde{\lambda}_2)\,|z_1|^\rho\,|\epsilon_1(\boldsymbol{x})|$, so that, for some positive constants A_1 and A_2,

$$\begin{aligned}\tau_2(z_1)\left(\tau_2(z_1)^{-1}\left|\epsilon_1(\boldsymbol{x})\right|\left\|\boldsymbol{z}_2\right\|_*\right)^{s_0} &\leq A_1\left|z_1\right|^{-\rho}\left|z_1\right|^{s_0\rho}\left|\epsilon_1(\boldsymbol{x})\right|^{s_0}\left\|\boldsymbol{z}_2\right\|_*^{s_0} \\ &= A_1\left|z_1\right|^{s_0\rho-\rho}\left|\epsilon_1(\boldsymbol{x})\right|^{s_0}\left\|\boldsymbol{z}_2\right\|_*^{\rho}\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} \\ &\leq A_2\left|\boldsymbol{x}\right|^{s_0\rho-\rho}\left|\epsilon_1(\boldsymbol{x})\right|^{s_0}\left|\boldsymbol{x}\right|^{\rho}\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} \\ &= A_2\left|\boldsymbol{x}\right|^{s_0\rho}\left|\epsilon_1(\boldsymbol{x})\right|^{s_0}\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} \\ &= o(1)\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0}.\end{aligned}$$

Here the first equation is again due to the definition of α and the last one follows because $|\epsilon_1(\boldsymbol{x})| = o(|\boldsymbol{x}|^{-\rho})$ by assumption. The second inequality follows because $|z_1| \leq |\boldsymbol{z}| \leq c\,|\boldsymbol{x}|$ and similarly with $|z_1|$ replaced by $|\boldsymbol{z}_2|$ (see footnote 7). Hence, as $\alpha s_0 = s_0 - \rho$, we find that, for $|z_1|$ large,

$$E\left[|\overline{g}(\boldsymbol{x}) + \varepsilon_1|^{s_0}\right] \leq |z_1|^{s_0} - \tfrac{1}{2}\tilde{\lambda}_2\,|z_1|^{\alpha s_0} + o(1)\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0}, \tag{48}$$

where $o(1) \to 0$ as $|\boldsymbol{x}| \to \infty$.

To combine the cases $s_0 < 1$ and $s_0 \geq 1$, set $\tilde{\lambda}_0 = \tilde{\lambda}_1 \wedge \frac{1}{2}\tilde{\lambda}_2 \in (0,1)$ and conclude from (46) and (48) that, for $|z_1|$ large,

$$E\left[|\overline{g}(\boldsymbol{x}) + \varepsilon_1|^{s_0}\right] \leq |z_1|^{s_0} - \tilde{\lambda}_0\,|z_1|^{\alpha s_0} + o(1)\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0}, \tag{49}$$

where $o(1) \to 0$ as $|\boldsymbol{x}| \to \infty$.

Step 3: Bounding (41). First conclude from inequalities (42) and (49) that, for $|z_1|$ large,

$$\begin{aligned}E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq 1 + |z_1|^{s_0} - \tilde{\lambda}_0\,|z_1|^{\alpha s_0} + \overline{s}_1\,|z_1|^{\alpha s_0} \\ + s_1\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} - \eta_0 s_1\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} + o(1)\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0}.\end{aligned}$$

Furthermore, we noted earlier that the value of s_1, and hence also the value of $\overline{s}_1$, can be chosen as close to zero as desired. Therefore, for $|\boldsymbol{z}_2|$ large enough and for some $\overline{\eta} \in (0,1)$,

$$-\eta_0 s_1\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} + o(1)\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} \leq -\overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} \leq -\overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0},$$

where the replacement of $\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0}$ with $\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0}$ is justified because $\alpha \in (0,1)$ (this replacement is needed below). Also, as we can assume that the value of $\overline{s}_1$ is so small that $\tilde{\lambda}_0 - \overline{s}_1 > 0$, we have $-\tilde{\lambda}_0\,|z_1|^{\alpha s_0} + \overline{s}_1\,|z_1|^{\alpha s_0} = -\overline{\lambda}\,|z_1|^{\alpha s_0}$ where $\overline{\lambda} \in (0,1)$ (the upper bound follows because $\tilde{\lambda}_0 < 1$, as noted above). Thus, we can conclude that, for $|z_1|$ and $|\boldsymbol{z}_2|$ large,

$$E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq 1 + |z_1|^{s_0} + s_1\left\|\boldsymbol{z}_2\right\|_*^{\alpha s_0} - \overline{\lambda}\,|z_1|^{\alpha s_0} - \overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0}.$$

Now, let $\left\|\boldsymbol{z}_2\right\|_*$ be so large that $\overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0} \geq c > 1$. Then,

$$-\overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0} = -1 - \overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0}\left(1 - 1/(\overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0})\right) \leq -1 - \overline{\eta}\,(1 - 1/c)\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0},$$

where $\overline{\eta}\,(1 - 1/c) \in (0,1]$ and, setting $\overline{c} = \overline{\lambda} \wedge (\overline{\eta}\,(1 - 1/c))$, we have $\overline{c} \in (0,1]$ and

$$-\overline{\lambda}\,|z_1|^{\alpha s_0} - \overline{\eta}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0} \leq -1 - \overline{c}\,|z_1|^{\alpha s_0} - \overline{c}\left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0} \leq -\overline{c}\left(1 + |z_1|^{\alpha s_0} + \left\|\boldsymbol{z}_2\right\|_*^{\alpha^2 s_0}\right).$$

Next note that

$$-\bar{c}\big(1+|z_1|^{\alpha s_0}+\|\boldsymbol{z}_2\|_*^{\alpha^2 s_0}\big) \le -\bar{c}\left(1+|z_1|^{s_0}+\|\boldsymbol{z}_2\|_*^{\alpha s_0}\right)^{\alpha} \le -\bar{c}\left(1+|z_1|^{s_0}+s_1\|\boldsymbol{z}_2\|_*^{\alpha s_0}\right)^{\alpha},$$

where the first inequality follows because $\alpha \in (0,1)$ and the second one because $s_1 < 1$ by assumption. This implies that

$$-\overline{\lambda}\,|z_1|^{\alpha s_0} - \overline{\eta}\,\|\boldsymbol{z}_2\|_*^{\alpha^2 s_0} \le -\bar{c}\left(1+|z_1|^{s_0}+s_1\|\boldsymbol{z}_2\|_*^{\alpha s_0}\right)^{\alpha}.$$

By the preceding discussion we can find positive (and finite) constants M_i and $\overline{M}_i$ $(i=1,2)$ such that

$$\begin{aligned} E\left[V(\boldsymbol{y}_1)\mid \boldsymbol{y}_0=\boldsymbol{x}\right] \le 1+|z_1|^{s_0}+s_1\|\boldsymbol{z}_2\|_*^{\alpha s_0} - \bar{c}\left(1+|z_1|^{s_0}+s_1\|\boldsymbol{z}_2\|_*^{\alpha s_0}\right)^{\alpha} \\ +\overline{M}_1\mathbf{1}_{C_1}(z_1)+\overline{M}_2\mathbf{1}_{C_2}(\boldsymbol{z}_2), \end{aligned} \tag{50}$$

where $C_1=\{z_1\in\mathbb{R} : |z_1|\le M_1\}$ and $C_2=\{\boldsymbol{z}_2\in\mathbb{R}^{p-1} : |\boldsymbol{z}_2|\le M_2\}$.

Step 4: Completing the proof. Using the definition $V(\boldsymbol{x})=1+|z_1(\boldsymbol{x})|^{s_0}+s_1\|\boldsymbol{z}_2(\boldsymbol{x})\|_*^{\alpha s_0}$ and letting $L\ge \overline{M}_1+\overline{M}_2$, we obtain from (50) that

$$E\left[V(\boldsymbol{y}_1)\mid \boldsymbol{y}_0=\boldsymbol{x}\right] \le V(\boldsymbol{x})-\bar{c}V(\boldsymbol{x})^{\alpha}+L=(1-h(\boldsymbol{x}))\,V(\boldsymbol{x})+L,$$

where $h(\boldsymbol{x})=\bar{c}V(\boldsymbol{x})^{\alpha-1}$.

As $\alpha\in(0,1)$ and $\bar{c}\in(0,1]$, we have $0<h(\boldsymbol{x})\le\bar{c}$ and $h(\boldsymbol{x})\to 0$, as $|\boldsymbol{x}|\to\infty$. Comparing the above inequality with inequality (37) (see the proof of Theorem 2 (Part (i), Step 4)) and the properties of the function $h(\boldsymbol{x})$ shows that we can verify Condition D with arguments similar to those in the aforementioned proof. Specifically, we need to show that $L<\frac{1}{2}h(\boldsymbol{x})\,(1-h(\boldsymbol{x}))^{\frac{1}{2}}\,V(\boldsymbol{x})$ holds for all $|\boldsymbol{x}|$ large enough. That this holds is seen by noting that (see the definition of $h(\boldsymbol{x})$ above)

$$\tfrac{1}{2}h(\boldsymbol{x})\,(1-h(\boldsymbol{x}))^{\frac{1}{2}}\,V(\boldsymbol{x})=\tfrac{1}{2}\bar{c}\left(1-\bar{c}V(\boldsymbol{x})^{\alpha-1}\right)^{\frac{1}{2}}V(\boldsymbol{x})^{\alpha},$$

where $V(\boldsymbol{x})^{\alpha}\to\infty$ and $V(\boldsymbol{x})^{\alpha-1}\to 0$, as $|\boldsymbol{x}|\to\infty$.

Hence, as in the proof of Theorem 2 (Part (i), Step 4) we can conclude that there exist positive constants M and b such that, for $C=\{\boldsymbol{x}\in\mathbb{R}^p : |\boldsymbol{x}|\le M\}$,

$$E\left[V(\boldsymbol{y}_1)\mid \boldsymbol{y}_0=\boldsymbol{x}\right]\le V(\boldsymbol{x})-\phi_1\left(V(\boldsymbol{x})\right)+b\mathbf{1}_C(\boldsymbol{x}),$$

where $\phi_1(v)=\frac{1}{2}h(\boldsymbol{x})V(\boldsymbol{x})=\frac{1}{2}\bar{c}v^{\alpha}$. This implies that Condition D holds with $\phi=\phi_1$. The result follows from Theorem 1 (note that $\alpha=1-\rho/s_0$ so that $1-\alpha=\rho/s_0$).

Case $p=1$: As in the corresponding proof of Theorem 2, we have $\boldsymbol{x}=x_1=u$, so we simply write x for any of these and note the following: Model (16) reduces to $y_t=y_{t-1}+\tilde{g}(y_{t-1})+\varepsilon_t$, Assumption 1(i) becomes redundant, Assumption 1(ii.a) is automatically satisfied with $g(x)=x+\tilde{g}(x)$, $\epsilon(x)=0$, and d redundant (as long as the condition $|g(x)|\to\infty$ as $|x|\to\infty$ still holds), and Assumptions 1(ii.b) and 2 are as when $p\ge 2$. In other words, the model can be written as $y_t=g(y_{t-1})+\varepsilon_t$ with g satisfying Assumption 1(ii.b) as well as $|g(x)|\to\infty$ as

$|x| \to \infty$. Note further that now $z_1(\boldsymbol{x})$ reduces to x_1 and we simply write x in place of either of these. Also, due to the choice $g(x) = x + \tilde{g}(x)$ we have $\overline{g}(\boldsymbol{x}) = g(x)$.

We go through the changes needed in the proof of Theorem 3 in case $p \geq 2$. Note that the equality $V(\boldsymbol{x}) = 1 + |z_1|^{s_0} + s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0}$ in case $p \geq 2$ reduces to $V(x) = 1 + |x|^{s_0}$ by setting $s_1 = 0$. The beginning of the proof until (41) remains valid with (41) reducing to

$$E\left[V(y_1) \mid y_0 = x\right] = 1 + E\left[|g(x) + \varepsilon_1|^{s_0}\right]. \tag{51}$$

Step 1 can be omitted as the term considered therein equals zero. In Step 2, setting $\epsilon(x) = 0$ inequalities (43) and (44) remain valid, and so does (45). The numbered inequalities (46)–(49) all hold but in all of them the last term is set to zero. In Step 3, the first inequality holds with s_1, $\bar{s}_1$, and the $o(1)$ term all set to zero. In the following arguments, set $\bar{\eta} = 0$ and $\bar{\lambda} = \tilde{\lambda}_0$. Now, some slight changes are needed. Set $\bar{c} = \bar{\lambda}/2 \in (0, 1)$ and assume $|x|$ is so large that $|x|^{\alpha s_0} \geq 1/\bar{c}$. This implies that

$$-\overline{\lambda}\,|x|^{\alpha s_0} \leq -1 - \bar{c}\,|x|^{\alpha s_0} \leq -\bar{c}\big(1 + |x|^{\alpha s_0}\big) \leq -\bar{c}\,(1 + |x|^{s_0})^{\alpha}$$

similarly to the corresponding derivations in Step 3. Therefore, inequality (50) holds with s_1 and $\overline{M}_2$ set to zero. Step 4 remains valid, so that the stated (f, r)-ergodicity result is obtained from Theorem 1 with $f = V^{1-\delta(1-\alpha)} = V^{1-\delta\rho/s_0} = (1 + |x|^{s_0})^{1-\delta\rho/s_0}$ and $\delta \in [1, 1/(1-\alpha)]$. Denoting, for brevity, $\gamma = 1 - \delta\rho/s_0 \in (0, 1]$ note that $1 + |x|^{s_0 - \delta\rho} = 1 + (|x|^{s_0})^{\gamma} = \{[1 + (|x|^{s_0})^{\gamma}]^{1/\gamma}\}^{\gamma} \leq \{C[1 + |x|^{s_0}]\}^{\gamma}$ for some finite positive C (due to Loève's c_r-inequality) so that the (f, r)-ergodicity with $f(x) = 1 + |x|^{s_0 - \delta\rho}$ follows. ■

Proof of Corollary to Theorem 3. First consider the case $p \geq 2$. We find from the proof of Theorem 3 (the beginning of Step 3) that, for $|z_1|$ large,

$$E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq 1 + |z_1|^{s_0} - (\tilde{\lambda}_0 - \overline{s}_1)\,|z_1|^{\alpha s_0} + s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - (\eta_0 s_1 - o(1))\,\|\boldsymbol{z}_2\|_*^{\alpha s_0},$$

where $\overline{s}_1$ is so small that $\tilde{\lambda}_0 - \overline{s}_1 > 0$ holds and $o(1) \to 0$ as $|\boldsymbol{x}| \to \infty$. Hence, defining $\overline{\eta} \in (0, 1)$, $\overline{M}_1 \mathbf{1}_{C_1}(z_1)$, and $\overline{M}_2 \mathbf{1}_{C_2}(\boldsymbol{z}_2)$ as in the proof of Theorem 3 (Step 3), we have

$$\begin{aligned} E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq 1 + |z_1|^{s_0} + s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - (\tilde{\lambda}_0 - \overline{s}_1)\,|z_1|^{\alpha s_0} - \overline{\eta}\,\|\boldsymbol{z}_2\|_*^{\alpha s_0} \\ + \overline{M}_1 \mathbf{1}_{C_1}(z_1) + \overline{M}_2 \mathbf{1}_{C_2}(\boldsymbol{z}_2), \end{aligned}$$

and setting $c_1 = (\tilde{\lambda}_0 - \overline{s}_1) \wedge \overline{\eta}$,

$$E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq 1 + |z_1|^{s_0} + s_1 \|\boldsymbol{z}_2\|_*^{\alpha s_0} - c_1\,(|z_1|^{\alpha s_0} + \|\boldsymbol{z}_2\|_*^{\alpha s_0}) + \overline{M}_1 \mathbf{1}_{C_1}(z_1) + \overline{M}_2 \mathbf{1}_{C_2}(\boldsymbol{z}_2).$$

As $V(\boldsymbol{x}) = 1 + |z_1(\boldsymbol{x})|^{s_0} + s_1 \|\boldsymbol{z}_2(\boldsymbol{x})\|_*^{\alpha s_0}$ and $\alpha s_0 = s_0 - \rho$ we can write this, for all $\boldsymbol{x}$, as

$$E\left[V(\boldsymbol{y}_1) \mid \boldsymbol{y}_0 = \boldsymbol{x}\right] \leq V(\boldsymbol{x}) - c_1\left(|z_1(\boldsymbol{x})|^{s_0-\rho} + \|\boldsymbol{z}_2(\boldsymbol{x})\|_*^{s_0-\rho}\right) + \overline{M}_1 \mathbf{1}_{C_1}(z_1(\boldsymbol{x})) + \overline{M}_2 \mathbf{1}_{C_2}(\boldsymbol{z}_2(\boldsymbol{x})).$$

From Theorem 14.3.7 of Meyn and Tweedie (2009) we now find that $\pi\left(|z_1(\boldsymbol{x})|^{s_0-\rho} + \|\boldsymbol{z}_2(\boldsymbol{x})\|_*^{s_0-\rho}\right) < \infty$ and, by the equivalence of vector norms in $\mathbb{R}^p$, $\pi\left(|z_1(\boldsymbol{x})|^{s_0-\rho} + |\boldsymbol{z}_2(\boldsymbol{x})|^{s_0-\rho}\right) < \infty$ also holds. Furthermore, as $|z_1(\boldsymbol{x})|^{s_0-\rho} + |\boldsymbol{z}_2(\boldsymbol{x})|^{s_0-\rho} \geq c_2\,(|z_1(\boldsymbol{x})| + |\boldsymbol{z}_2(\boldsymbol{x})|)^{s_0-\rho} \geq c_2\,|\boldsymbol{z}(\boldsymbol{x})|^{s_0-\rho}$

and $|\boldsymbol{z}(\boldsymbol{x})|^{s_0-\rho} = |\boldsymbol{A}\boldsymbol{x}|^{s_0-\rho} \geq c_3 |\boldsymbol{x}|^{s_0-\rho}$ for some c_2, $c_3 \in (0, \infty)$ (that depend on s_0 and ρ), it follows that $\pi\left(|\boldsymbol{x}|^{s_0-\rho}\right) < \infty$.

In the case $p = 1$, the above arguments hold if one sets $\overline{s}_1 = 0$, $c_1 = \tilde{\lambda}_0$, and drops all the terms related to $\boldsymbol{z}_2$. ■

Additional references

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