Supplementary Appendix to 'Testing for observation-dependent regime switching in mixture autoregressive models' by Meitz and Saikkonen.

D Further details for the general results

Proof of Lemma A.1. Note that $\sup_{\alpha\in A}\|(\hat{\beta}_{T\alpha},\hat{\pi}_{T\alpha},\hat{\varpi}_{T\alpha})-(\beta^*,\pi^*,0)\| \leq \sup_{\alpha\in A}\|\hat{\beta}_{T\alpha}-\beta^*\|+\sup_{\alpha\in A}\|(\hat{\pi}_{T\alpha},\hat{\varpi}_{T\alpha})-(\pi^*,0)\|$. The former term on the majorant side is $o_p(1)$ by Assumption 2(ii). The latter term equals $\sup_{\alpha\in A}\|\boldsymbol{\pi}_\alpha(\hat{\phi}_{T\alpha},\hat{\varphi}_{T\alpha})-\boldsymbol{\pi}_\alpha(\phi^*,\phi^*)\|$ which, due to the assumptions made, can be bounded by $C\sup_{\alpha\in A}h\big(\|(\hat{\phi}_{T\alpha},\hat{\varphi}_{T\alpha})-(\phi^*,\phi^*)\|_*\big) \leq Ch\big(\sup_{\alpha\in A}\|(\hat{\phi}_{T\alpha},\hat{\varphi}_{T\alpha})-(\phi^*,\phi^*)\|_*\big)$. By Assumption 2(ii) and the equivalence of all vector norms on $\mathbb{R}^{2q_2}$, the majorant side is $o_p(1)$. ■

Proof of Lemma 1, further details. To justify that the last term on the right hand side of (28) is dominated by $-\frac{1}{4}\|\boldsymbol{\theta}_{T\alpha}\|^2+o_{p\alpha}(1)$, note first that

$$\left(o_{p\alpha}(1)-\frac{1}{2}\right)\|\boldsymbol{\theta}_{T\alpha}\|^2=\left(o_{p\alpha}(1)-\frac{1}{4}\right)\|\boldsymbol{\theta}_{T\alpha}\|^2-\frac{1}{4}\|\boldsymbol{\theta}_{T\alpha}\|^2:=W_{T\alpha}\|\boldsymbol{\theta}_{T\alpha}\|^2-\frac{1}{4}\|\boldsymbol{\theta}_{T\alpha}\|^2,$$

where $W_{T\alpha}=-\frac{1}{4}+o_{p\alpha}(1)$. Thus, $P\left(\sup_{\alpha\in A}W_{T\alpha}\leq 0\right)\to 1$ and (here $\mathbf{1}(\cdot)$ denotes the indicator function)

$$\begin{aligned}\sup_{\alpha\in A}W_{T\alpha}\|\boldsymbol{\theta}_{T\alpha}\|^2&=\sup_{\alpha\in A}W_{T\alpha}\|\boldsymbol{\theta}_{T\alpha}\|^2\,\mathbf{1}\left(\sup_{\alpha\in A}W_{T\alpha}\leq 0\right)+\sup_{\alpha\in A}W_{T\alpha}\|\boldsymbol{\theta}_{T\alpha}\|^2\,\mathbf{1}\left(\sup_{\alpha\in A}W_{T\alpha}>0\right)\\&\leq\sup_{\alpha\in A}W_{T\alpha}\|\boldsymbol{\theta}_{T\alpha}\|^2\,\mathbf{1}\left(\sup_{\alpha\in A}W_{T\alpha}>0\right),\end{aligned}$$

where the last term is non-negative and positive with probability that is at most $P\left(\sup_{\alpha\in A}W_{T\alpha}>0\right)\to 0$. Thus, combining the above derivations yields the desired result $\left(o_{p\alpha}(1)-\frac{1}{2}\right)\|\boldsymbol{\theta}_{T\alpha}\|^2\leq-\frac{1}{4}\|\boldsymbol{\theta}_{T\alpha}\|^2+o_{p\alpha}(1)$.

To justify the use of the continuous mapping theorem, note that in the first instance it is applied with the function $g:\mathcal{C}(A,\mathbb{R}^r)\times\{\mathcal{I}_\alpha\}\to\mathcal{C}(A,\mathbb{R}^r)$ mapping $(x_\bullet,\mathcal{I}_\bullet)$ to $\mathcal{I}_\bullet^{-1}x_\bullet$. Here $\mathcal{I}_\alpha^{-1}x_\alpha$ is continuous in α by Assumption 5(iii). Also, the latter set in the product $\mathcal{C}(A,\mathbb{R}^r)\times\{\mathcal{I}_\alpha\}$ contains only the non-random function $\mathcal{I}_\alpha$; this product space can be equipped with essentially the same metric as $\mathcal{C}(A,\mathbb{R}^r)$; cf. Andrews and Ploberger (1994, p. 1392 and 1407) and Zhu and Zhang (2006, proof of Theorem 5). In the second instance, the continuous mapping theorem is applied with the function $g:\mathcal{B}(A,\mathbb{R}^r)\to\mathbb{R}$ mapping $x_\bullet$ ($\in\mathcal{B}(A,\mathbb{R}^r)$) to $\sup_{\alpha\in A}\|x_\alpha\|$. For continuity, we need to establish that if a sequence $x_{n\bullet}$ converges to $x_\bullet$ in $\mathcal{B}(A,\mathbb{R}^r)$, then $g(x_{n\bullet})\to g(x_\bullet)$ in $\mathbb{R}$ (i.e., if $\sup_{\alpha\in A}\|x_{n\alpha}-x_\alpha\|\to 0$, then $|\sup_{\alpha\in A}\|x_{n\alpha}\|-\sup_{\alpha\in A}\|x_\alpha\||\to 0$). The triangle inequality implies that $\sup_{\alpha\in A}\|x_{n\alpha}\|\leq\sup_{\alpha\in A}\|x_{n\alpha}-x_\alpha\|+\sup_{\alpha\in A}\|x_\alpha\|$, as well as the same result with $x_{n\alpha}$ and x_α interchanged, and the desired result follows from these inequalities. ■

Proof of Lemma 4, further details. It remains to verify the continuity mentioned in the proof. For simplicity, consider the continuity of the functions $g_1:\mathcal{C}(A,\mathbb{R}^r)\times\{\mathcal{I}_\alpha\}\to\mathcal{B}(A,\mathbb{R})$ mapping $(x_\bullet,\mathcal{I}_\bullet)$ to $x_\bullet'\mathcal{I}_\bullet x_\bullet$ and $g_2:\mathcal{C}(A,\mathbb{R}^r)\times\{\mathcal{I}_\alpha\}\to\mathcal{B}(A,\mathbb{R})$ mapping $(x_\bullet,\mathcal{I}_\bullet)$ to $\inf_{\boldsymbol{\lambda}\in\Lambda}\{(\boldsymbol{\lambda}-x_\bullet)'\mathcal{I}_\bullet(\boldsymbol{\lambda}-x_\bullet)\}$ separately. For g_1, continuity is rather clear, for if a sequence $(x_{n\bullet},\mathcal{I}_\bullet)$ converges to $(x_\bullet,\mathcal{I}_\bullet)$ in $\mathcal{C}(A,\mathbb{R}^r)\times\{\mathcal{I}_\alpha\}$, then $g_1((x_{n\bullet},\mathcal{I}_\bullet))\to g_1((x_\bullet,\mathcal{I}_\bullet))$ in $\mathcal{B}(A,\mathbb{R})$ (i.e., if $\sup_{\alpha\in A}\|x_{n\alpha}-x_\alpha\|\to 0$, then

$\sup_{\alpha\in A}|x_{n\alpha}'\mathcal{I}_\alpha x_{n\alpha}-x_\alpha'\mathcal{I}_\alpha x_\alpha|\to 0$). For the continuity of g_2, suppose that $\sup_{\alpha\in A}\|x_{n\alpha}-x_\alpha\|\to 0$, and consider $\sup_{\alpha\in A}|\inf_{\boldsymbol{\lambda}\in\Lambda}\{(\boldsymbol{\lambda}-x_{n\alpha})'\mathcal{I}_\alpha(\boldsymbol{\lambda}-x_{n\alpha})\}-\inf_{\boldsymbol{\lambda}\in\Lambda}\{(\boldsymbol{\lambda}-x_\alpha)'\mathcal{I}_\alpha(\boldsymbol{\lambda}-x_\alpha)\}|$. Noting that

$$\inf_{\boldsymbol{\lambda}\in\Lambda}\left\{(\boldsymbol{\lambda}-x_{n\alpha})'\mathcal{I}_\alpha(\boldsymbol{\lambda}-x_{n\alpha})\right\}=\left\{\inf_{\boldsymbol{\lambda}\in\Lambda}\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_{n\alpha})\|\right\}^2$$

and similarly for the other infimum, we need to consider

$$\sup_{\alpha\in A}\left\{\left|\inf_{\boldsymbol{\lambda}\in\Lambda}\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_{n\alpha})\|-\inf_{\boldsymbol{\lambda}\in\Lambda}\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_\alpha)\|\right|\left(\inf_{\boldsymbol{\lambda}\in\Lambda}\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_{n\alpha})\|+\inf_{\boldsymbol{\lambda}\in\Lambda}\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_\alpha)\|\right)\right\}. \quad (45)$$

Using the triangle inequality and properties of the Euclidean vector norm,

$$\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_{n\alpha})\|\le\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_\alpha)\|+\|\mathcal{I}_\alpha^{1/2}(x_{n\alpha}-x_\alpha)\|\le\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_\alpha)\|+\left(\lambda_{\max}\left(\mathcal{I}_\alpha\right)\right)^{1/2}\|x_{n\alpha}-x_\alpha\|,$$

and similarly with $x_{n\alpha}$ and x_α exchanged, so that

$$\left|\inf_{\boldsymbol{\lambda}\in\Lambda}\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_{n\alpha})\|-\inf_{\boldsymbol{\lambda}\in\Lambda}\|\mathcal{I}_\alpha^{1/2}(\boldsymbol{\lambda}-x_\alpha)\|\right|\le\left(\lambda_{\max}\left(\mathcal{I}_\alpha\right)\right)^{1/2}\|x_{n\alpha}-x_\alpha\|.$$

As was noted after Assumption 6, the cone Λ contains the origin, so that the term in (45) in parentheses is dominated by $\left(\lambda_{\max}\left(\mathcal{I}_\alpha\right)\right)^{1/2}\left(\|x_{n\alpha}\|+\|x_\alpha\|\right)$. Now, due to Assumption 5(iii), the fact that $x_{n\bullet}, x_\bullet$ are bounded, and the assumed $\sup_{\alpha\in A}\|x_{n\alpha}-x_\alpha\|\to 0$, the quantity in (45) converges to zero. ■

Proof of Lemma 5. For brevity and clarity, within this proof we use somewhat simplified notation and let

$$\mathcal{I}_\alpha^{-1}=\begin{bmatrix}A & B\\ B' & C\end{bmatrix}$$

denote the partition of $\mathcal{I}_\alpha^{-1}$ (so that, e.g., C is shorthand for $(\mathcal{I}_\alpha^{-1})_{\vartheta\vartheta}$). This implies that $\mathcal{I}_\alpha$ can be expressed as

$$\mathcal{I}_\alpha=\begin{bmatrix}D^{-1} & -D^{-1}BC^{-1}\\ -C^{-1}B'D^{-1} & C^{-1}+C^{-1}B'D^{-1}BC^{-1}\end{bmatrix}$$

where $D=A-BC^{-1}B'$ (thus, e.g., $D^{-1}=\mathcal{I}_{\theta\theta\alpha}$). Note also that A, C, and D are symmetric (as $\mathcal{I}_\alpha$ is symmetric).

First note that $S_\alpha=\mathcal{I}_\alpha Z_\alpha$ can be expressed as

$$S_\alpha=\begin{bmatrix}D^{-1} & -D^{-1}BC^{-1}\\ -C^{-1}B'D^{-1} & C^{-1}+C^{-1}B'D^{-1}BC^{-1}\end{bmatrix}\begin{bmatrix}Z_{\theta\alpha}\\ Z_{\vartheta\alpha}\end{bmatrix}=\begin{bmatrix}D^{-1}Z_{\theta\alpha}-D^{-1}BC^{-1}Z_{\vartheta\alpha}\\ -C^{-1}B'D^{-1}Z_{\theta\alpha}+C^{-1}Z_{\vartheta\alpha}+C^{-1}B'D^{-1}BC^{-1}Z_{\vartheta\alpha}\end{bmatrix}$$

so that $S_{\theta\alpha}'DS_{\theta\alpha}$ equals

$$\begin{aligned}S_{\theta\alpha}'DS_{\theta\alpha}&=(D^{-1}Z_{\theta\alpha}-D^{-1}BC^{-1}Z_{\vartheta\alpha})'D(D^{-1}Z_{\theta\alpha}-D^{-1}BC^{-1}Z_{\vartheta\alpha})\\&=Z_{\theta\alpha}'D^{-1}Z_{\theta\alpha}-Z_{\theta\alpha}'D^{-1}BC^{-1}Z_{\vartheta\alpha}-Z_{\vartheta\alpha}'C^{-1}B'D^{-1}Z_{\theta\alpha}+Z_{\vartheta\alpha}'C^{-1}B'D^{-1}BC^{-1}Z_{\vartheta\alpha}.\end{aligned}$$

Now, since $Z_\alpha'\mathcal{I}_\alpha Z_\alpha$ can be written as

$$\begin{aligned}Z_\alpha'\mathcal{I}_\alpha Z_\alpha&=\begin{bmatrix}Z_{\theta\alpha}\\ Z_{\vartheta\alpha}\end{bmatrix}'\begin{bmatrix}D^{-1} & -D^{-1}BC^{-1}\\ -C^{-1}B'D^{-1} & C^{-1}+C^{-1}B'D^{-1}BC^{-1}\end{bmatrix}\begin{bmatrix}Z_{\theta\alpha}\\ Z_{\vartheta\alpha}\end{bmatrix}\\&=Z_{\theta\alpha}'D^{-1}Z_{\theta\alpha}-Z_{\theta\alpha}'D^{-1}BC^{-1}Z_{\vartheta\alpha}-Z_{\vartheta\alpha}'C^{-1}B'D^{-1}Z_{\theta\alpha}\\&\quad+Z_{\vartheta\alpha}'C^{-1}Z_{\vartheta\alpha}+Z_{\vartheta\alpha}'C^{-1}B'D^{-1}BC^{-1}Z_{\vartheta\alpha},\end{aligned}$$

we obtain

$$Z_\alpha'\mathcal{I}_\alpha Z_\alpha = Z_{\vartheta\alpha}'C^{-1}Z_{\vartheta\alpha} + S_{\theta\alpha}'DS_{\theta\alpha}. \tag{46}$$

Now consider $\inf_{\boldsymbol{\lambda}\in\Lambda}\{(\boldsymbol{\lambda}-Z_\alpha)'\mathcal{I}_\alpha(\boldsymbol{\lambda}-Z_\alpha)\}$. Similarly as above,

$$\begin{aligned}
&(\boldsymbol{\lambda}-Z_\alpha)'\mathcal{I}_\alpha(\boldsymbol{\lambda}-Z_\alpha)\\
&=(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})'D^{-1}(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})-(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})'D^{-1}BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})-(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})'C^{-1}B'D^{-1}(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})\\
&+(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})'C^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})+(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})'C^{-1}B'D^{-1}BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})\\
&=(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})'D^{-1}(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})-(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})'D^{-1}[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]-[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]'D^{-1}(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})\\
&+(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})'C^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})+[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]'D^{-1}[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]\\
&=(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})'C^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})+\{(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})-[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]\}'D^{-1}\{(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})-[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]\}.
\end{aligned}$$

Now, for any fixed $\boldsymbol{\lambda}_\vartheta\in\mathbb{R}^{q_\vartheta}$, Assumption 7 implies that

$$\inf_{\boldsymbol{\lambda}_\theta\in\mathbb{R}^{q_\theta}}\{(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})-[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]\}'D^{-1}\{(\boldsymbol{\lambda}_\theta-Z_{\theta\alpha})-[BC^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})]\}=0$$

(cf. Andrews (1999, eqn. (7.35))) so that

$$\inf_{\boldsymbol{\lambda}\in\Lambda}\left\{(\boldsymbol{\lambda}-Z_\alpha)'\mathcal{I}_\alpha(\boldsymbol{\lambda}-Z_\alpha)\right\}=\inf_{\boldsymbol{\lambda}_\vartheta\in\Lambda_\vartheta}\left\{(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})'C^{-1}(\boldsymbol{\lambda}_\vartheta-Z_{\vartheta\alpha})\right\}. \tag{47}$$

Combining (46) and (47) and recalling that $C^{-1}=(\mathcal{I}_\alpha^{-1})_{\vartheta\vartheta}^{-1}$ and $D=\mathcal{I}_{\theta\theta\alpha}^{-1}$ yields the equality stated in the lemma.

Finally, $(\mathcal{I}_\alpha^{-1})_{\vartheta\vartheta}$ and $Z_{\vartheta\alpha}$ can be expressed as

$$\begin{aligned}
(\mathcal{I}_\alpha^{-1})_{\vartheta\vartheta}&=(\mathcal{I}_{\vartheta\vartheta\alpha}-\mathcal{I}_{\vartheta\theta\alpha}\mathcal{I}_{\theta\theta\alpha}^{-1}\mathcal{I}_{\theta\vartheta\alpha})^{-1}\ [=\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}+\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}\mathcal{I}_{\vartheta\theta\alpha}(\mathcal{I}_{\theta\theta\alpha}-\mathcal{I}_{\theta\vartheta\alpha}\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}\mathcal{I}_{\vartheta\theta\alpha})^{-1}\mathcal{I}_{\theta\vartheta\alpha}\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}],\\
Z_{\vartheta\alpha}&=(\mathcal{I}_\alpha^{-1})_{\vartheta\vartheta}(S_{\vartheta\alpha}-\mathcal{I}_{\vartheta\theta\alpha}\mathcal{I}_{\theta\theta\alpha}^{-1}S_{\theta\alpha})\ [=\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}S_{\vartheta\alpha}+\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}\mathcal{I}_{\vartheta\theta\alpha}(\mathcal{I}_{\theta\theta\alpha}-\mathcal{I}_{\theta\vartheta\alpha}\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}\mathcal{I}_{\vartheta\theta\alpha})^{-1}(\mathcal{I}_{\theta\vartheta\alpha}\mathcal{I}_{\vartheta\vartheta\alpha}^{-1}S_{\vartheta\alpha}-S_{\theta\alpha})],
\end{aligned}$$

where the two different expressions result from two different ways of writing the inverse of a partitioned matrix. ■

Proof of Lemma 6. The required arguments are standard but presented for completeness and to contrast them with arguments that lead to Lemma 4. The reparameterization described in Assumption 3 is unnecessary and the original $\tilde{\phi}$–parameterization may be used (alternatively, consider the identity mapping $\pi=\boldsymbol{\pi}(\tilde{\phi})=\tilde{\phi}$). As for the quadratic expansion of the log-likelihood function, let $\boldsymbol{\theta}(\tilde{\phi})=(\tilde{\phi}-\tilde{\phi}^*)$ take the role of $\boldsymbol{\theta}(\alpha,\beta,\pi,\varpi)$, and note that straightforward derivations (similar to those used in the LMAR example in Section 3.3.1) yield

$$\begin{aligned}
L_T^0(\tilde{\phi})-L_T^0(\tilde{\phi}^*)&=(T^{-1/2}S_T^0)'[T^{1/2}\boldsymbol{\theta}(\tilde{\phi})]-\tfrac{1}{2}[T^{1/2}\boldsymbol{\theta}(\tilde{\phi})]'\mathcal{I}^0[T^{1/2}\boldsymbol{\theta}(\tilde{\phi})]+R_T(\tilde{\phi}),\\
R_T(\tilde{\phi})&=\tfrac{1}{2}[T^{1/2}\boldsymbol{\theta}(\tilde{\phi})]'[T^{-1}\nabla_{\phi\phi'}L_T^0(\dot{\phi})-(-\mathcal{I}^0)][T^{1/2}\boldsymbol{\theta}(\tilde{\phi})],
\end{aligned}$$

with $\dot{\phi}$ denoting a point between $\tilde{\phi}$ and $\tilde{\phi}^*$. Validity of Assumption 5 follows from the arguments used in connection with the LMAR example together with Assumption 8. Assumption 6 holds with $\Lambda=\mathbb{R}^{p+2}$. Arguments analogous to those that lead to Lemma 4 now yield the stated convergence result, and the convergence is joint as in both cases it follows from the weak convergence result $T^{-1/2}S_{T\bullet}\Rightarrow S_\bullet$. ■

E Further details for the LMAR example

E.1 Verification of Assumption 5(ii), further details

As for the weak convergence requirement in part (ii), we rely on Theorem 2 (and the remarks that follow it) in Andrews and Ploberger (1995). As can be seen from the proof of their Theorem 2, it suffices to verify their conditions EP1(a), EP1(e), and EP4 (omitting the weakly exogeneous X_t variables therein). Under the null hypothesis, y_t is a linear Gaussian AR(p) process so that condition EP1(a) is satisfied with geometrically declining mixing numbers. To check condition EP1(e), we show that $E\big[\sup_{\alpha\in A}\sup_{(\pi,\varpi)\in\Pi}|l_t^{\pi}(\alpha,\pi,\varpi)|\big]<\infty$, $E\big[\sup_{\alpha\in A}\|\nabla_{(\pi,\varpi)}l_t^{\pi}(\alpha,\pi^*,0)\|^r\big]<\infty$ for any positive r, and $E\big[\sup_{\alpha\in A}\sup_{(\pi,\varpi)\in\Pi}\|\nabla^2_{(\pi,\varpi)(\pi,\varpi)'}l_t^{\pi}(\alpha,\pi,\varpi)\|\big]<\infty$. The first of these moment conditions follows from the arguments used to verify our Assumption 2 (see the verification of this Assumption for the GMAR model; the details for the LMAR model are presented there). The second holds due to the expression of $\nabla_{(\pi,\varpi)}l_t^{\pi}(\alpha,\pi^*,0)$ in (29), the fact that $0<\alpha_{1,t}^L(\alpha)<1$, and Lemma F.1. The third is verified below in Supplementary Appedix E.2. As the compactness requirement of condition EP4(a) holds by our Assumption 1(ii), it remains to verify EP4(b). To this end, note that for arbitrary $a,b\in A$,

$$\|\nabla_{(\pi,\varpi)}l_t^{\pi}(a,\pi^*,0)-\nabla_{(\pi,\varpi)}l_t^{\pi}(b,\pi^*,0)\|^r=|\alpha_{1,t}^L(a)-\alpha_{1,t}^L(b)|^r\left\|\frac{\nabla f_t(\pi^*)}{f_t(\pi^*)}\right\|^r.$$

By straightforward differentiation, $\nabla_{\alpha}\alpha_{1,t}^L(\alpha)=\alpha_{1,t}^L(\alpha)(1-\alpha_{1,t}^L(\alpha))(1,y_{t-1},\ldots,y_{t-m})$, so that by the mean value theorem

$$\alpha_{1,t}^L(a)-\alpha_{1,t}^L(b)=\alpha_{1,t}^L(c_{a,b})(1-\alpha_{1,t}^L(c_{a,b}))(1,y_{t-1},\ldots,y_{t-m})'(a-b)$$

for some $c_{a,b}\in\mathbb{R}^{m+1}$ between a and b (as A is not necessarily convex, $c_{a,b}$ does not necessarily belong to A, but this has no effect in what follows as the expression $\alpha_{1,t}^L(c_{a,b})$ is nevertheless well defined for all $c_{a,b}\in\mathbb{R}^{m+1}$). Setting $B_t=1+|y_{t-1}|+\ldots+|y_{t-m}|$ and noting that $0<\alpha_{1,t}^L(c_{a,b})<1$ this implies that

$$|\alpha_{1,t}^L(a)-\alpha_{1,t}^L(b)|\le|(1,y_{t-1},\ldots,y_{t-m})'(a-b)|\le(1+|y_{t-1}|+\ldots+|y_{t-m}|)\|a-b\|=B_t\|a-b\|.$$

Hence

$$E\Big[\sup_{a,b\in A,\|a-b\|<\delta}\|\nabla_{(\pi,\varpi)}l_t^{\pi}(a,\pi^*,0)-\nabla_{(\pi,\varpi)}l_t^{\pi}(b,\pi^*,0)\|^r\Big]<\delta^rE\left[B_t^r\left\|\frac{\nabla f_t(\pi^*)}{f_t(\pi^*)}\right\|^r\right]$$

where on the majorant side the expectation is finite (due to the fact that the y_t's possess moments of all orders, see also the proof of Lemma F.1). Hence condition EP4(b) holds, and the desired weak convergence follows.

E.2 Verification of Assumption 5(iv), further details

It remains to show that $E\big[\sup_{\alpha\in A}\sup_{(\pi,\varpi)\in\Pi}\|\nabla^2_{(\pi,\varpi)(\pi,\varpi)'}l_t^{\pi}(\alpha,\pi,\varpi)\|\big]<\infty$. This in turn follows if we show the same with $\nabla^2_{(\pi,\varpi)(\pi,\varpi)'}l_t^{\pi}(\alpha,\pi,\varpi)$ replaced by $\nabla^2_{\pi\pi'}l_t^{\pi}(\alpha,\pi,\varpi)$, $\nabla^2_{\varpi\varpi'}l_t^{\pi}(\alpha,\pi,\varpi)$, and $\nabla^2_{\pi\varpi'}l_t^{\pi}(\alpha,\pi,\varpi)$. Consider the expression of $\nabla^2_{\pi\pi'}l_t^{\pi}(\alpha,\pi,\varpi)$ given in Appendix B and recall that $0<\alpha_{1,t}^L(\alpha),\alpha_{2,t}^L(\alpha)<1$ and $\nabla f_t(\pi)=f_t(\pi)\nabla_{\pi}l_t^0(\pi)$ with $l_t^0(\pi)=\log[f_t(\pi)]$. Then we can, for instance, write

$$\left\|\frac{\alpha_{1,t}^{L}(\alpha)\nabla f_t(\pi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)}\right\| = \left\|\frac{\alpha_{1,t}^{L}(\alpha)f_t(\pi)\nabla_\pi l_t^0(\pi)}{\alpha_{1,t}^{L}(\alpha)f_t(\pi)+\alpha_{2,t}^{L}(\alpha)f_t(\pi-\varpi)}\right\| \leq \left\|\nabla_\pi l_t^0(\pi)\right\|$$

and

$$\begin{aligned}\left\|\frac{\alpha_{1,t}^{L}(\alpha)\nabla' f_t(\pi)+\alpha_{2,t}^{L}(\alpha)\nabla' f_t(\pi-\varpi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)}\right\| &= \left\|\frac{\alpha_{1,t}^{L}(\alpha)f_t(\pi)\nabla_\pi l_t^0(\pi)+\alpha_{2,t}^{L}(\alpha)f_t(\pi-\varpi)\nabla_\pi l_t^0(\pi-\varpi)}{\alpha_{1,t}^{L}(\alpha)f_t(\pi)+\alpha_{2,t}^{L}(\alpha)f_t(\pi-\varpi)}\right\| \\ &\leq \left\|\nabla_\pi l_t^0(\pi)\right\| + \left\|\nabla_\pi l_t^0(\pi-\varpi)\right\|.\end{aligned}$$

As similar inequalities can be obtained for the second term of $\nabla_{\pi\pi'}^2 l_t^\pi(\alpha,\pi,\varpi)$, we get

$$\begin{aligned}\left\|\nabla_{\pi\pi'}^2 l_t^\pi(\alpha,\pi,\varpi)\right\| \leq{}& \left\|\alpha_{1,t}^{L}(\alpha)\frac{\nabla^2 f_t(\pi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)}\right\| + \left\|\nabla_\pi l_t^0(\pi)\right\|^2 + \left\|\nabla_\pi l_t^0(\pi)\right\|\left\|\nabla_\pi l_t^0(\pi-\varpi)\right\| \\ &+ \left\|\alpha_{2,t}^{L}(\alpha)\frac{\nabla^2 f_t(\pi-\varpi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)}\right\| + \left\|\nabla_\pi l_t^0(\pi-\varpi)\right\|^2 + \left\|\nabla_\pi l_t^0(\pi)\right\|\left\|\nabla_\pi l_t^0(\pi-\varpi)\right\|.\end{aligned}$$

Next note that

$$\frac{\nabla^2 f_t(\pi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)} = \frac{\nabla(f_t(\pi)\nabla_\pi l_t^0(\pi))}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)} = \frac{f_t(\pi)\nabla_\pi l_t^0(\pi)\nabla_{\pi'} l_t^0(\pi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)} + \frac{f_t(\pi)\nabla_{\pi\pi'}^2 l_t^0(\pi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)},$$

so that arguments similar to those already used above give

$$\left\|\alpha_{1,t}^{L}(\alpha)\frac{\nabla^2 f_t(\pi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)}\right\| \leq \left\|\nabla_\pi l_t^0(\pi)\right\|^2 + \left\|\nabla_{\pi\pi'}^2 l_t^0(\pi)\right\|$$

and

$$\left\|\alpha_{2,t}^{L}(\alpha)\frac{\nabla^2 f_t(\pi-\varpi)}{f_{2,t}^{\pi}(\alpha,\pi,\varpi)}\right\| \leq \left\|\nabla_\pi l_t^0(\pi-\varpi)\right\|^2 + \left\|\nabla_{\pi\pi'}^2 l_t^0(\pi-\varpi)\right\|.$$

Hence, we can conclude that

$$\begin{aligned}\left\|\nabla_{\pi\pi'}^2 l_t^\pi(\alpha,\pi,\varpi)\right\| \leq{}& 2\left\|\nabla_\pi l_t^0(\pi)\right\|^2 + 2\left\|\nabla_\pi l_t^0(\pi-\varpi)\right\|^2 + 2\left\|\nabla_\pi l_t^0(\pi)\right\|\left\|\nabla_\pi l_t^0(\pi-\varpi)\right\| \\ &+\left\|\nabla_{\pi\pi'}^2 l_t^0(\pi)\right\| + \left\|\nabla_{\pi\pi'}^2 l_t^0(\pi-\varpi)\right\|.\end{aligned}$$

To bound the expression on the dominant side, note that $\nabla_\pi l_t^0(\pi) = \frac{\nabla f_t(\pi)}{f_t(\pi)}$ and $\nabla_{\pi\pi'}^2 l_t^0(\pi) = \frac{\nabla^2 f_t(\pi)}{f_t(\pi)} - \frac{\nabla f_t(\pi)}{f_t(\pi)}\frac{\nabla' f_t(\pi)}{f_t(\pi)}$ so that Lemma F.1 ensures that $E\left[\sup_{\alpha\in A}\sup_{(\pi,\varpi)\in\Pi}\left\|\nabla_{\pi\pi'}^2 l_t^\pi(\alpha,\pi,\varpi)\right\|\right] < \infty$. An inspection of the expressions of $\nabla_{\pi\varpi'}^2 l_t^\pi(\alpha,\pi,\varpi)$ and $\nabla_{\varpi\varpi'}^2 l_t^\pi(\alpha,\pi,\varpi)$ in Appendix B shows that a similar result can be obtained with $\nabla_{\pi\pi'}^2 l_t^\pi(\alpha,\pi,\varpi)$ replaced by $\nabla_{\pi\varpi'}^2 l_t^\pi(\alpha,\pi,\varpi)$ and $\nabla_{\varpi\varpi'}^2 l_t^\pi(\alpha,\pi,\varpi)$, yielding the desired result.

F Further details for the GMAR example

F.1 Partial derivatives of the reparameterized log-likelihood function

Here we present certain partial derivatives of $l_t^{\pi}(\alpha,\beta,\pi,\varpi)$ with respect to (β,π,ϖ). For brevity, set $\tilde{\pi}=(\beta,\pi)$ (and similarly $\tilde{\pi}^*=(\beta^*,\pi^*)$), so that the desired derivatives are with respect to $\tilde{\pi}$ and ϖ or, elementwise, with respect to $\tilde{\pi}_i$ and ϖ_j for $i=1,\ldots,p+2$ and $j=1,\ldots,p+1$. In the derivative expressions below, the subindices in $\tilde{\pi}$ and ϖ are tacitly assumed to be withing these ranges. For brevity, denote $l_t^{\pi*}=l_t^{\pi}(\alpha,\tilde{\pi}^*,0)$, $f_t^*=f_t(\tilde{\pi}^*)$, $\mathsf{n}_p^*=\mathsf{n}_p(\tilde{\pi}^*)$, and similarly for their partial derivatives.

The following derivatives are obtained with straightforward (but tedious and lengthy) differentiation. The necessary calculations for the first- and second-order derivatives are presented in Supplementary Appendix F.7, but for brevity we omit the detailed calculations for the third- and fourth-order derivatives.

First- and second-order derivatives:

$$\begin{aligned}
\nabla_{\tilde{\pi}_i} l_t^{\pi*} &= \frac{\nabla_i f_t^*}{f_t^*}\\
\nabla_{\varpi_j} l_t^{\pi*} &= 0\\
\nabla^2_{\tilde{\pi}_i\tilde{\pi}_j} l_t^{\pi*} &= \frac{\nabla^2_{ij} f_t^*}{f_t^*} - \frac{\nabla_i f_t^*}{f_t^*}\frac{\nabla_j f_t^*}{f_t^*}.\\
\nabla^2_{\tilde{\pi}_i\varpi_j} l_t^{\pi*} &= 0\\
\nabla^2_{\varpi_i\varpi_j} l_t^{\pi*} &= \alpha_1\alpha_2\left[\frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{j+1} f_t^*}{f_t^*} + \frac{\nabla_{i+1} f_t^*}{f_t^*}\frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla^2_{i+1,j+1} f_t^*}{f_t^*}\right]
\end{aligned}$$

Third-order derivatives:

$$\begin{aligned}
\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k} l_t^{\pi*} &= \frac{\nabla^3_{ijk} f_t^*}{f_t^*} - \frac{\nabla^2_{ij} f_t^*}{f_t^*}\frac{\nabla_k f_t^*}{f_t^*} - \frac{\nabla^2_{ik} f_t^*}{f_t^*}\frac{\nabla_j f_t^*}{f_t^*} - \frac{\nabla^2_{jk} f_t^*}{f_t^*}\frac{\nabla_i f_t^*}{f_t^*} + 2\frac{\nabla_i f_t^*}{f_t^*}\frac{\nabla_j f_t^*}{f_t^*}\frac{\nabla_k f_t^*}{f_t^*}\\
\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\varpi_k} l_t^{\pi*} &= -\alpha_1\alpha_2\frac{\nabla_i f_t^*}{f_t^*}\left(\frac{\nabla^2_{j,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} - \frac{\nabla_j\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\right)\\
\nabla^3_{\tilde{\pi}_i\varpi_j\varpi_k} l_t^{\pi*} &= \alpha_1\alpha_2\Bigg[\frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\left(\frac{\nabla^2_{i,j+1} f_t^*}{f_t^*} - \frac{\nabla_i f_t^*}{f_t^*}\frac{\nabla_{j+1} f_t^*}{f_t^*}\right) + \frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\left(\frac{\nabla^2_{i,k+1} f_t^*}{f_t^*} - \frac{\nabla_i f_t^*}{f_t^*}\frac{\nabla_{k+1} f_t^*}{f_t^*}\right)\\
&\quad + \left(\frac{\nabla^3_{i,j+1,k+1} f_t^*}{f_t^*} - \frac{\nabla_i f_t^*}{f_t^*}\frac{\nabla^2_{j+1,k+1} f_t^*}{f_t^*}\right) + \left(\frac{\nabla^2_{i,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} - \frac{\nabla_i\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\right)\frac{\nabla_{j+1} f_t^*}{f_t^*}\\
&\quad + \left(\frac{\nabla^2_{i,j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} - \frac{\nabla_i\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\right)\frac{\nabla_{k+1} f_t^*}{f_t^*}\Bigg]\\
\nabla^3_{\varpi_i\varpi_j\varpi_k} l_t^{\pi*} &= \alpha_1\alpha_2(\alpha_2-\alpha_1)\Bigg[\frac{\nabla^2_{i+1,j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{k+1} f_t^*}{f_t^*} + \frac{\nabla^2_{i+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{j+1} f_t^*}{f_t^*} + \frac{\nabla^2_{j+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{i+1} f_t^*}{f_t^*}\\
&\quad + \frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{j+1,k+1} f_t^*}{f_t^*} + \frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,k+1} f_t^*}{f_t^*} + \frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,j+1} f_t^*}{f_t^*}\\
&\quad + \frac{\nabla^3_{i+1,j+1,k+1} f_t^*}{f_t^*}\Bigg]
\end{aligned}$$

Fourth-order derivative (fourth-order derivatives with respect to $\tilde{\pi}$ will not be explicitly needed, and thus we omit their expressions):

$$
\begin{aligned}
&\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l} l_t^{\pi *} \\
&= -\alpha_1^2\alpha_2^2\Bigg[\left(\frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{j+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\right)\frac{\nabla_{l+1}f_t^*}{f_t^*} \\
&\qquad + \left(\frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{j+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\right)\frac{\nabla_{k+1}f_t^*}{f_t^*} \\
&\qquad + \left(\frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{k+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\right)\frac{\nabla_{j+1}f_t^*}{f_t^*} \\
&\qquad + \left(\frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{k+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{j+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*} + \frac{\nabla_{l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{j+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\right)\frac{\nabla_{i+1}f_t^*}{f_t^*} \\
&\qquad + \left(\frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{j+1}f_t^*}{f_t^*} + \frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{i+1}f_t^*}{f_t^*} + \frac{\nabla^2_{i+1,j+1}f_t^*}{f_t^*}\right) \\
&\qquad\qquad \times \left(\frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{l+1}f_t^*}{f_t^*} + \frac{\nabla_{l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{k+1}f_t^*}{f_t^*} + \frac{\nabla^2_{k+1,l+1}f_t^*}{f_t^*}\right) \\
&\qquad + \left(\frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{k+1}f_t^*}{f_t^*} + \frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{i+1}f_t^*}{f_t^*} + \frac{\nabla^2_{i+1,k+1}f_t^*}{f_t^*}\right) \\
&\qquad\qquad \times \left(\frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{l+1}f_t^*}{f_t^*} + \frac{\nabla_{l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{j+1}f_t^*}{f_t^*} + \frac{\nabla^2_{j+1,l+1}f_t^*}{f_t^*}\right) \\
&\qquad + \left(\frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{l+1}f_t^*}{f_t^*} + \frac{\nabla_{l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{i+1}f_t^*}{f_t^*} + \frac{\nabla^2_{i+1,l+1}f_t^*}{f_t^*}\right) \\
&\qquad\qquad \times \left(\frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{k+1}f_t^*}{f_t^*} + \frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{j+1}f_t^*}{f_t^*} + \frac{\nabla^2_{j+1,k+1}f_t^*}{f_t^*}\right)\Bigg] \\
&+ \alpha_1\alpha_2(1-3\alpha_1\alpha_2)\Bigg[\frac{\nabla^3_{i+1,j+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{l+1}f_t^*}{f_t^*} + \frac{\nabla^3_{i+1,j+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{k+1}f_t^*}{f_t^*} + \frac{\nabla^3_{i+1,k+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{j+1}f_t^*}{f_t^*} \\
&\qquad + \frac{\nabla^3_{j+1,k+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla_{i+1}f_t^*}{f_t^*} + \frac{\nabla_{i+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^3_{j+1,k+1,l+1}f_t^*}{f_t^*} + \frac{\nabla_{j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^3_{i+1,k+1,l+1}f_t^*}{f_t^*} \\
&\qquad + \frac{\nabla_{k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^3_{i+1,j+1,l+1}f_t^*}{f_t^*} + \frac{\nabla_{l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^3_{i+1,j+1,k+1}f_t^*}{f_t^*} + \frac{\nabla^4_{i+1,j+1,k+1,l+1}f_t^*}{f_t^*}\Bigg] \\
&+ \alpha_1\alpha_2(\alpha_2-\alpha_1)^2\Bigg[\frac{\nabla^2_{i+1,j+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{k+1,l+1}f_t^*}{f_t^*} + \frac{\nabla^2_{i+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{j+1,l+1}f_t^*}{f_t^*} + \frac{\nabla^2_{i+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{j+1,k+1}f_t^*}{f_t^*} \\
&\qquad + \frac{\nabla^2_{j+1,k+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,l+1}f_t^*}{f_t^*} + \frac{\nabla^2_{j+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,k+1}f_t^*}{f_t^*} + \frac{\nabla^2_{k+1,l+1}\mathsf{n}_p^*}{\mathsf{n}_p^*}\frac{\nabla^2_{i+1,j+1}f_t^*}{f_t^*}\Bigg]
\end{aligned}
$$

F.2 Fourth-order expansion of the log-likelihood function

Justification of (38) and expression of $R_T^{(1)}(\alpha,\tilde{\pi},\varpi)$. Straightforward calculation yields the fourth-order Taylor expansion (38) with the remainder term (for brevity, we again write $\tilde{\pi}=(\beta,\pi)$ and $\tilde{\pi}^*=(\beta^*,\pi^*)$)

$$\begin{aligned}
R_T^{(1)}(\alpha,\tilde{\pi},\varpi) &= \varpi'\nabla_{\varpi}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)+\frac{2}{2!}(\tilde{\pi}-\tilde{\pi}^*)'\nabla^2_{\tilde{\pi}\varpi'}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)\varpi \\
&+\frac{1}{3!}\sum_{i=1}^{q_1+q_2}\sum_{j=1}^{q_1+q_2}\sum_{k=1}^{q_1+q_2}\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)(\tilde{\pi}_i-\tilde{\pi}_i^*)(\tilde{\pi}_j-\tilde{\pi}_j^*)(\tilde{\pi}_k-\tilde{\pi}_k^*) \\
&+\frac{3}{3!}\sum_{i=1}^{q_1+q_2}\sum_{j=1}^{q_1+q_2}\sum_{k=1}^{q_2}\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\varpi_k}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)(\tilde{\pi}_i-\tilde{\pi}_i^*)(\tilde{\pi}_j-\tilde{\pi}_j^*)\varpi_k \\
&+\frac{1}{3!}\sum_{i=1}^{q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\nabla^3_{\varpi_i\varpi_j\varpi_k}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)\varpi_i\varpi_j\varpi_k \\
&+\frac{1}{4!}\sum_{i=1}^{q_1+q_2}\sum_{j=1}^{q_1+q_2}\sum_{k=1}^{q_1+q_2}\sum_{l=1}^{q_1+q_2}\nabla^4_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k\tilde{\pi}_l}L_T^{\pi}(\alpha,\dot{\tilde{\pi}},\dot{\varpi})(\tilde{\pi}_i-\tilde{\pi}_i^*)(\tilde{\pi}_j-\tilde{\pi}_j^*)(\tilde{\pi}_k-\tilde{\pi}_k^*)(\tilde{\pi}_l-\tilde{\pi}_l^*) \\
&+\frac{4}{4!}\sum_{i=1}^{q_1+q_2}\sum_{j=1}^{q_1+q_2}\sum_{k=1}^{q_1+q_2}\sum_{l=1}^{q_2}\nabla^4_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k\varpi_l}L_T^{\pi}(\alpha,\dot{\tilde{\pi}},\dot{\varpi})(\tilde{\pi}_i-\tilde{\pi}_i^*)(\tilde{\pi}_j-\tilde{\pi}_j^*)(\tilde{\pi}_k-\tilde{\pi}_k^*)\varpi_l \\
&+\frac{6}{4!}\sum_{i=1}^{q_1+q_2}\sum_{j=1}^{q_1+q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2}\nabla^4_{\tilde{\pi}_i\tilde{\pi}_j\varpi_k\varpi_l}L_T^{\pi}(\alpha,\dot{\tilde{\pi}},\dot{\varpi})(\tilde{\pi}_i-\tilde{\pi}_i^*)(\tilde{\pi}_j-\tilde{\pi}_j^*)\varpi_k\varpi_l \\
&+\frac{4}{4!}\sum_{i=1}^{q_1+q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2}\nabla^4_{\tilde{\pi}_i\varpi_j\varpi_k\varpi_l}L_T^{\pi}(\alpha,\dot{\tilde{\pi}},\dot{\varpi})(\tilde{\pi}_i-\tilde{\pi}_i^*)\varpi_j\varpi_k\varpi_l \\
&+\frac{1}{4!}\sum_{i=1}^{q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2}\big(\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^{\pi}(\alpha,\dot{\tilde{\pi}},\dot{\varpi})-\nabla_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)\big)\varpi_i\varpi_j\varpi_k\varpi_l, \quad (48)
\end{aligned}$$

where $(\dot{\tilde{\pi}},\dot{\varpi})$ denotes a point between $(\tilde{\pi},\varpi)$ and $(\tilde{\pi}^*,0)$.

Justification of (39) and expression of $R_T(\alpha,\tilde{\pi},\varpi)$. We begin with some useful notation. Let $\mathfrak{I}$ denote the index set

$$\mathfrak{I}=((1,1),(2,2),\ldots,(q_2,q_2),(1,2),(1,3),\ldots,(1,q_2),(2,3),\ldots,(q_2-1,q_2)).$$

For any scalars (or $d\times 1$ column vectors) A_{ij} indexed by i and j (here and elsewhere it is tacitly assumed these indices belong to $\{1,\ldots,q_2\}$), let $[A_{ij}]_{(i,j)\in\mathfrak{I}}$ denote the following $1\times q_2(q_2+1)/2$ row vector (or $d\times q_2(q_2+1)/2$ matrix):

$$[A_{ij}]_{(i,j)\in\mathfrak{I}}=[A_{11}:\cdots:A_{q_2q_2}:A_{12}:\cdots:A_{q_2-1,q_2}]\,.$$

For instance, $v(\varpi)=[\varpi_i\varpi_j]'_{(i,j)\in\mathfrak{I}}$. Similarly, for any scalars A_{ijkl} indexed by i,j,k,l, let $[A_{ijkl}]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}}$

denote the following $q_2(q_2+1)/2 \times q_2(q_2+1)/2$ matrix

$$[A_{ijkl}]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}} = \begin{bmatrix} A_{1111} & \cdots & A_{11q_2q_2} & A_{1112} & \cdots & A_{1,1,q_2-1,q_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{q_2q_211} & \cdots & A_{q_2q_2q_2q_2} & A_{q_2q_212} & \cdots & A_{q_2,q_2,q_2-1,q_2} \\ A_{1211} & \cdots & A_{12q_2q_2} & A_{1212} & \cdots & A_{1,2,q_2-1,q_2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{q_2-1,q_2,1,1} & \cdots & A_{q_2-1,q_2,q_2,q_2} & A_{q_2-1,q_2,1,2} & \cdots & A_{q_2-1,q_2,q_2-1,q_2} \end{bmatrix}.$$

With this notation, and for any scalars A_{ijkl} and B_{ij} such that $A_{ijkl} = A_{jikl}$, $A_{ijkl} = A_{ijlk}$, and $B_{ij} = B_{ji}$ for all i,j,k,l, it holds that[10]

$$[B_{ij}]_{(i,j)\in\mathfrak{I}}\,[c_{ij}c_{kl}A_{ijkl}]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}}\,[B_{kl}]'_{(k,l)\in\mathfrak{I}} = \frac{1}{4}\sum_{i=1}^{q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2} A_{ijkl}B_{ij}B_{kl}, \tag{49}$$

where the c_{ij}'s are as in Section 3.3.1 ($c_{ij} = 1/2$ if $i = j$ and $c_{ij} = 1$ if $i \neq j$).

Now, to obtain (39), introduce the matrix

$$\mathcal{J}_T = \begin{bmatrix} \mathcal{J}_{T,\tilde{\pi}\tilde{\pi}} & \mathcal{J}'_{T,\tilde{\pi}\varpi\varpi} \\ \mathcal{J}_{T,\tilde{\pi}\varpi\varpi} & \mathcal{J}_{T,\varpi\varpi\varpi\varpi} \end{bmatrix}$$

where the matrices $\mathcal{J}_{T,\tilde{\pi}\tilde{\pi}}$ $((q_1+q_2)\times(q_1+q_2))$, $\mathcal{J}'_{T,\tilde{\pi}\varpi\varpi}$ $((q_1+q_2)\times q_\vartheta)$, and $\mathcal{J}_{T,\varpi\varpi\varpi\varpi}$ $(q_\vartheta \times q_\vartheta)$ are defined as follows (here $\nabla^2_{\tilde{\pi}\tilde{\pi}'}L_T^{\pi*}$ stands for $\nabla^2_{\tilde{\pi}\tilde{\pi}'}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)$ etc.)

$$\begin{aligned}
\mathcal{J}_{T,\tilde{\pi}\tilde{\pi}} &= -T^{-1}\nabla^2_{\tilde{\pi}\tilde{\pi}'}L_T^{\pi*}, \\
\mathcal{J}'_{T,\tilde{\pi}\varpi\varpi} &= -T^{-1}\frac{1}{\alpha_1\alpha_2}\left[c_{ij}\nabla^3_{\tilde{\pi}\varpi_i\varpi_j}L_T^{\pi*}\right]_{(i,j)\in\mathfrak{I}} \\
&= -T^{-1}\frac{1}{\alpha_1\alpha_2}[c_{11}\nabla^3_{\tilde{\pi}\varpi_1\varpi_1}L_T^{\pi*} : \cdots : c_{q_2-1,q_2}\nabla^3_{\tilde{\pi}\varpi_{q_2-1}\varpi_{q_2}}L_T^{\pi*}], \\
\mathcal{J}_{T,\varpi\varpi\varpi\varpi} &= -T^{-1}\frac{8}{4!}\frac{1}{\alpha_1^2\alpha_2^2}\left[c_{ij}c_{kl}\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^{\pi*}\right]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}} \\
&= -T^{-1}\frac{8}{4!}\frac{1}{\alpha_1^2\alpha_2^2}\begin{bmatrix} c_{11}c_{11}\nabla^4_{\varpi_1\varpi_1\varpi_1\varpi_1}L_T^{\pi*} & \cdots & c_{q_2-1,q_2}c_{11}\nabla^4_{\varpi_{q_2-1}\varpi_{q_2}\varpi_1\varpi_1}L_T^{\pi*} \\ \vdots & \ddots & \vdots \\ c_{11}c_{q_2-1,q_2}\nabla^4_{\varpi_1\varpi_1\varpi_{q_2-1}\varpi_{q_2}}L_T^{\pi*} & \cdots & c_{q_2-1,q_2}c_{q_2-1,q_2}\nabla^4_{\varpi_{q_2-1}\varpi_{q_2}\varpi_{q_2-1}\varpi_{q_2}}L_T^{\pi*} \end{bmatrix}.
\end{aligned}$$

---

[10] To justify (49), partition the index set $\mathfrak{I}$ as $\mathfrak{I} = (\mathfrak{I}_1, \mathfrak{I}_2)$ with $\mathfrak{I}_1 = ((1,1),(2,2),\ldots(q_2,q_2))$ and $\mathfrak{I}_2 = ((1,2),(1,3),\ldots,(1,q_2),(2,3),\ldots,(q_2-1,q_2))$. With straightforward algebra,

$$\begin{aligned}
&[B_{ij}]_{(i,j)\in\mathfrak{I}}\,[c_{ij}c_{kl}A_{ijkl}]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}}\,[B_{kl}]'_{(k,l)\in\mathfrak{I}} \\
&= \sum_{(i,j)\in\mathfrak{I}}\sum_{(k,l)\in\mathfrak{I}} c_{ij}c_{kl}A_{ijkl}B_{ij}B_{kl} \\
&= \frac{1}{4}\sum_{(i,j)\in\mathfrak{I}_1}\sum_{(k,l)\in\mathfrak{I}_1} A_{ijkl}B_{ij}B_{kl} + \frac{1}{2}\sum_{(i,j)\in\mathfrak{I}_1}\sum_{(k,l)\in\mathfrak{I}_2} A_{ijkl}B_{ij}B_{kl} + \frac{1}{2}\sum_{(i,j)\in\mathfrak{I}_2}\sum_{(k,l)\in\mathfrak{I}_1} A_{ijkl}B_{ij}B_{kl} + \sum_{(i,j)\in\mathfrak{I}_2}\sum_{(k,l)\in\mathfrak{I}_2} A_{ijkl}B_{ij}B_{kl} \\
&= \frac{1}{4}\left[\sum_{i=j}\sum_{k=l} A_{ijkl}B_{ij}B_{kl} + 2\sum_{i=j}\sum_{k<l} A_{ijkl}B_{ij}B_{kl} + 2\sum_{i<j}\sum_{k=l} A_{ijkl}B_{ij}B_{kl} + 4\sum_{i<j}\sum_{k<l} A_{ijkl}B_{ij}B_{kl}\right] \\
&= \frac{1}{4}\sum_{i=1}^{q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2} A_{ijkl}B_{ij}B_{kl},
\end{aligned}$$

where the properties $A_{ijkl} = A_{jikl}$, $A_{ijkl} = A_{ijlk}$, and $B_{ij} = B_{ji}$ for all i,j,k,l, are used in the last equality.

Straightforward computations (for the third one, use property (49)) now show that

$$-\frac{1}{2}T^{1/2}(\tilde{\pi}-\tilde{\pi}^*)'\mathcal{J}_{T,\tilde{\pi}\tilde{\pi}}T^{1/2}(\tilde{\pi}-\tilde{\pi}^*) = \frac{1}{2}(\tilde{\pi}-\tilde{\pi}^*)'\nabla^2_{\tilde{\pi}\tilde{\pi}'}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)(\tilde{\pi}-\tilde{\pi}^*),$$

$$\begin{aligned}
&-T^{1/2}(\tilde{\pi}-\tilde{\pi}^*)'\mathcal{J}'_{T,\tilde{\pi}\varpi\varpi}T^{1/2}\alpha_1\alpha_2 v(\varpi)\\
&\quad= (\tilde{\pi}-\tilde{\pi}^*)'\left[c_{11}\nabla^3_{\tilde{\pi}\varpi_1\varpi_1}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)\varpi_1^2 + ... + c_{q_2-1,q_2}\nabla^3_{\tilde{\pi}\varpi_{q_2-1}\varpi_{q_2}}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)\varpi_{q_2-1}\varpi_{q_2}\right]\\
&\quad= \frac{3}{3!}\sum_{i=1}^{q}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\nabla^3_{\tilde{\pi}_i\varpi_j\varpi_k}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)(\tilde{\pi}_i-\tilde{\pi}_i^*)\varpi_j\varpi_k,
\end{aligned}$$

$$-\frac{1}{2}T\alpha_1^2\alpha_2^2 v(\varpi)'\mathcal{J}_{T,\varpi\varpi\varpi\varpi}v(\varpi) = \frac{1}{4!}\sum_{i=1}^{q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2}\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^{\pi}(\alpha,\tilde{\pi}^*,0)\varpi_i\varpi_j\varpi_k\varpi_l.$$

Therefore the fourth-order Taylor expansion of $L_T^{\pi}(\alpha,\tilde{\pi},\varpi)$ in (38) can be written as a quadratic expansion given by

$$L_T^{\pi}(\alpha,\tilde{\pi},\varpi)-L_T^{\pi}(\alpha,\tilde{\pi}^*,0) = S_T'\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)-\frac{1}{2}[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]'\mathcal{J}_T[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]+R_T^{(1)}(\alpha,\tilde{\pi},\varpi). \quad (50)$$

Next, define

$$\mathcal{I} = \begin{bmatrix}\mathcal{I}_{\tilde{\pi}\tilde{\pi}} & \mathcal{I}'_{\tilde{\pi}\varpi\varpi}\\ \mathcal{I}_{\tilde{\pi}\varpi\varpi} & \mathcal{I}_{\varpi\varpi\varpi\varpi}\end{bmatrix}, \quad (51)$$

where the matrices $\mathcal{I}_{\tilde{\pi}\tilde{\pi}}$ $((q_1+q_2)\times(q_1+q_2))$, $\mathcal{I}'_{\tilde{\pi}\varpi\varpi}$ $((q_1+q_2)\times q_\vartheta)$, and $\mathcal{I}_{\varpi\varpi\varpi\varpi}$ $(q_\vartheta\times q_\vartheta)$ are defined as follows

$$\begin{aligned}
\mathcal{I}_{\tilde{\pi}\tilde{\pi}} &= E\left[\frac{\nabla f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)}\frac{\nabla' f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)}\right],\\
\mathcal{I}'_{\tilde{\pi}\varpi\varpi} &= \left[c_{ij}E\left[\frac{\nabla f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)}X^*_{t,i,j}\right]\right]_{(i,j)\in\mathfrak{I}},\\
\mathcal{I}_{\varpi\varpi\varpi\varpi} &= \left[c_{ij}c_{kl}E\left[X^*_{t,i,j}X^*_{t,k,l}\right]\right]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}},
\end{aligned}$$

and where we have used the short-hand notation $X^*_{t,i,j}$, $i,j\in\{1,\ldots,q_2\}$ (see (44)). Finiteness of $\mathcal{I}$ follows from Lemma F.1. Now, defining

$$R_T^{(2)}(\alpha,\tilde{\pi},\varpi) = -\frac{1}{2}[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]'(\mathcal{J}_T-\mathcal{I})[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]$$

and adding and subtracting terms, expansion (50) can be written as (39) with

$$R_T(\alpha,\tilde{\pi},\varpi) = R_T^{(1)}(\alpha,\tilde{\pi},\varpi)+R_T^{(2)}(\alpha,\tilde{\pi},\varpi).$$

F.3 Some more explicit derivatives and the verification of Assumption 5(iii)

Some more explicit derivative expressions. We will require more explicit expressions for the components of s_t in (43) (see also (40)–(42) and (44)). Straightforward computation shows that (as before, $\nabla f_t(\cdot)$ denotes differentiation of $f_t(\cdot)$ in (3) with respect to $\tilde{\phi} = (\tilde{\phi}_0,\tilde{\phi}_1,\ldots,\tilde{\phi}_p,\tilde{\sigma}_1^2)$)

$$\frac{\nabla f_t(\tilde{\phi})}{f_t(\tilde{\phi})} = \nabla\log(f_t(\tilde{\phi})), \qquad \frac{\nabla^2 f_t(\tilde{\phi})}{f_t(\tilde{\phi})} = \nabla^2\log(f_t(\tilde{\phi}))+\nabla\log(f_t(\tilde{\phi}))\nabla'\log(f_t(\tilde{\phi})),$$

where (as $\log(f_t(\tilde\phi)) = -\frac{1}{2}\log(2\bar\pi) - \frac{1}{2}\log(\tilde\sigma_1^2) - \frac{1}{2}g_t^2(\tilde\phi)$ with $g_t(\tilde\phi) = [y_t - (\tilde\phi_0 + \tilde\phi_1 y_{t-1} + \cdots + \tilde\phi_p y_{t-p})]/\tilde\sigma_1$)

$$\nabla \log(f_t(\tilde\phi)) = \begin{bmatrix} \frac{1}{\tilde\sigma_1} g_t(\tilde\phi) \\ \frac{1}{\tilde\sigma_1}\boldsymbol{y}_{t-1} g_t(\tilde\phi) \\ \frac{1}{2\tilde\sigma_1^2}(g_t^2(\tilde\phi)-1) \end{bmatrix}, \qquad \nabla^2 \log(f_t(\tilde\phi)) = \begin{bmatrix} -\frac{1}{\tilde\sigma_1^2} & -\frac{1}{\tilde\sigma_1^2}\boldsymbol{y}'_{t-1} & -\frac{1}{\tilde\sigma_1^3} g_t(\tilde\phi) \\ -\frac{1}{\tilde\sigma_1^2}\boldsymbol{y}_{t-1} & -\frac{1}{\tilde\sigma_1^2}\boldsymbol{y}_{t-1}\boldsymbol{y}'_{t-1} & -\frac{1}{\tilde\sigma_1^3} g_t(\tilde\phi)\boldsymbol{y}_{t-1} \\ -\frac{1}{\tilde\sigma_1^3} g_t(\tilde\phi) & -\frac{1}{\tilde\sigma_1^3} g_t(\tilde\phi)\boldsymbol{y}'_{t-1} & -\frac{1}{2\tilde\sigma_1^4}(2g_t^2(\tilde\phi)-1) \end{bmatrix},$$

so that

$$\frac{\nabla f_t(\pi^*)}{f_t(\pi^*)} = \begin{bmatrix} \frac{1}{\tilde\sigma_1^*}\varepsilon_t \\ \frac{1}{\tilde\sigma_1^*}\boldsymbol{y}_{t-1}\varepsilon_t \\ \frac{1}{2\tilde\sigma_1^{*2}}(\varepsilon_t^2-1) \end{bmatrix},$$

$$\frac{\nabla^2 f_t(\pi^*)}{f_t(\pi^*)} = \begin{bmatrix} -\frac{1}{\tilde\sigma_1^{*2}} & -\frac{1}{\tilde\sigma_1^{*2}}\boldsymbol{y}'_{t-1} & -\frac{1}{\tilde\sigma_1^{*3}}\varepsilon_t \\ -\frac{1}{\tilde\sigma_1^{*2}}\boldsymbol{y}_{t-1} & -\frac{1}{\tilde\sigma_1^{*2}}\boldsymbol{y}_{t-1}\boldsymbol{y}'_{t-1} & -\frac{1}{\tilde\sigma_1^{*3}}\boldsymbol{y}_{t-1}\varepsilon_t \\ -\frac{1}{\tilde\sigma_1^{*3}}\varepsilon_t & -\frac{1}{\tilde\sigma_1^{*3}}\boldsymbol{y}'_{t-1}\varepsilon_t & -\frac{1}{2\tilde\sigma_1^{*4}}(2\varepsilon_t^2-1) \end{bmatrix} + \begin{bmatrix} \frac{1}{\tilde\sigma_1^*}\varepsilon_t \\ \frac{1}{\tilde\sigma_1^*}\boldsymbol{y}_{t-1}\varepsilon_t \\ \frac{1}{2\tilde\sigma_1^{*2}}(\varepsilon_t^2-1) \end{bmatrix} \begin{bmatrix} \frac{1}{\tilde\sigma_1^*}\varepsilon_t \\ \frac{1}{\tilde\sigma_1^*}\boldsymbol{y}_{t-1}\varepsilon_t \\ \frac{1}{2\tilde\sigma_1^{*2}}(\varepsilon_t^2-1) \end{bmatrix}'$$

$$= \begin{bmatrix} \frac{1}{\tilde\sigma_1^{*2}}(\varepsilon_t^2-1) & \frac{1}{\tilde\sigma_1^{*2}}\boldsymbol{y}'_{t-1}(\varepsilon_t^2-1) & \frac{1}{2\tilde\sigma_1^{*3}}(\varepsilon_t^3-3\varepsilon_t) \\ \frac{1}{\tilde\sigma_1^{*2}}\boldsymbol{y}_{t-1}(\varepsilon_t^2-1) & \frac{1}{\tilde\sigma_1^{*2}}\boldsymbol{y}_{t-1}\boldsymbol{y}'_{t-1}(\varepsilon_t^2-1) & \frac{1}{2\tilde\sigma_1^{*3}}\boldsymbol{y}_{t-1}(\varepsilon_t^3-3\varepsilon_t) \\ \frac{1}{2\tilde\sigma_1^{*3}}(\varepsilon_t^3-3\varepsilon_t) & \frac{1}{2\tilde\sigma_1^{*3}}\boldsymbol{y}'_{t-1}(\varepsilon_t^3-3\varepsilon_t) & \frac{1}{4\tilde\sigma_1^{*4}}(\varepsilon_t^4-6\varepsilon_t^2+3) \end{bmatrix}. \tag{52}$$

Similar formulas hold for $\mathsf{n}_p(\boldsymbol{y}_{t-1};\tilde\phi)$. As $\log(\mathsf{n}_p(\boldsymbol{y}_{t-1};\tilde\phi)) = -\frac{p}{2}\log(2\bar\pi) - \frac{1}{2}\log(\det(\boldsymbol{\Gamma}_{1,p})) - \frac{1}{2}(\boldsymbol{y}_{t-1} - \mu_1\mathbf{1}_p)'\boldsymbol{\Gamma}_{1,p}^{-1}(\boldsymbol{y}_{t-1} - \mu_1\mathbf{1}_p)$ (see Section 2.2), we obtain, for each $i = 1,\dots,p+2$, (cf. Magnus and Neudecker (1999, p. 325))[11]

$$\nabla_i \log(\mathsf{n}_p(\boldsymbol{y}_{t-1};\tilde\phi)) = \frac{1}{2}tr\left(\frac{\partial\boldsymbol{\Gamma}_{1,p}^{-1}}{\partial\tilde\phi_i}\boldsymbol{\Gamma}_{1,p}\right) + (\boldsymbol{y}_{t-1}-\mu_1\mathbf{1}_p)'\boldsymbol{\Gamma}_{1,p}^{-1}(\frac{\partial\mu_1}{\partial\tilde\phi_i}\mathbf{1}_p) - \frac{1}{2}(\boldsymbol{y}_{t-1}-\mu_1\mathbf{1}_p)'\frac{\partial\boldsymbol{\Gamma}_{1,p}^{-1}}{\partial\tilde\phi_i}(\boldsymbol{y}_{t-1}-\mu_1\mathbf{1}_p),$$

where (see Section 2.2 for the notation)

$$\frac{\partial\mu_1}{\partial\tilde\phi} = \begin{bmatrix} (\tilde\phi(1))^{-1} \\ \tilde\phi_0(\tilde\phi(1))^{-2}\mathbf{1}_p \\ 0 \end{bmatrix}.$$

For the expression of $\frac{\partial\boldsymbol{\Gamma}_{1,p}^{-1}}{\partial\tilde\phi_i}$, first note that $\boldsymbol{\Gamma}_{1,p}^{-1}$ can be expressed as (see, e.g., Galbraith and Galbraith (1974)) $\boldsymbol{\Gamma}_{1,p}^{-1} = \frac{1}{\tilde\sigma_1^2}(U'U - V'V)$ with U and V being $p\times p$ Toeplitz matrices given by

$$U = \begin{bmatrix} 1 & & & \\ -\tilde\phi_1 & \ddots & & \\ \vdots & \ddots & \ddots & \\ -\tilde\phi_{p-1} & \cdots & -\tilde\phi_1 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} \tilde\phi_p & & & \\ \tilde\phi_{p-1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ \tilde\phi_1 & \cdots & \tilde\phi_{p-1} & \tilde\phi_p \end{bmatrix}.$$

Thus $\frac{\partial\boldsymbol{\Gamma}_{1,p}^{-1}}{\partial\tilde\phi_i}$ equals a zero matrix when differentiating with respect to $\tilde\phi_0$, $-\frac{1}{\tilde\sigma_1^2}\boldsymbol{\Gamma}_{1,p}^{-1}$ when differentiating with respect to $\tilde\sigma_1^2$, and

$$\frac{\partial\boldsymbol{\Gamma}_{1,p}^{-1}}{\partial\tilde\phi_i} = \frac{1}{\tilde\sigma_1^2}(\frac{\partial U'}{\partial\tilde\phi_i}U + U'\frac{\partial U}{\partial\tilde\phi_i} - \frac{\partial V'}{\partial\tilde\phi_i}V - V'\frac{\partial V}{\partial\tilde\phi_i})$$

[11] As before, ∇ denotes differentiation with respect to $\tilde\phi = (\tilde\phi_0, \tilde\phi_1, \dots, \tilde\phi_p, \tilde\sigma_1^2)$ and ∇_i, $i = 1,\dots,p+2$, with respect to the ith component of $\tilde\phi$.

when differentiating with respect to the autoregressive parameters. To summarize,

$$\frac{\nabla \mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} = \nabla \log(\mathsf{n}_p(\boldsymbol{y}_{t-1};\tilde{\phi})) = \begin{bmatrix} d_1(\boldsymbol{y}_{t-1};\tilde{\phi}) \\ d_2(\boldsymbol{y}_{t-1};\tilde{\phi}) \\ d_3(\boldsymbol{y}_{t-1};\tilde{\phi}) \end{bmatrix}, \tag{53}$$

where (note that $tr\left(\frac{\partial \boldsymbol{\Gamma}_{1,p}^{-1}}{\partial \tilde{\phi}_i}\boldsymbol{\Gamma}_{1,p}\right) = \frac{\partial vec(\boldsymbol{\Gamma}_{1,p}^{-1})'}{\partial \tilde{\phi}_i} vec(\boldsymbol{\Gamma}_{1,p}))$

$$\begin{aligned}
d_1(\boldsymbol{y}_{t-1};\tilde{\phi}) &= (\tilde{\phi}(1))^{-1}(\boldsymbol{y}_{t-1} - \mu_1 \mathbf{1}_p)'\boldsymbol{\Gamma}_{1,p}^{-1}\mathbf{1}_p, \\
d_2(\boldsymbol{y}_{t-1};\tilde{\phi}) &= \frac{1}{2}\frac{\partial vec(\boldsymbol{\Gamma}_{1,p}^{-1})'}{\partial(\tilde{\phi}_1,\ldots,\tilde{\phi}_p)} vec(\boldsymbol{\Gamma}_{1,p}) + \tilde{\phi}_0(\tilde{\phi}(1))^{-2}\mathbf{1}_p(\mathbf{1}_p'\boldsymbol{\Gamma}_{1,p}^{-1}(\boldsymbol{y}_{t-1} - \mu_1\mathbf{1}_p)) \\
&\quad - \frac{1}{2}\frac{\partial vec(\boldsymbol{\Gamma}_{1,p}^{-1})'}{\partial(\tilde{\phi}_1,\ldots,\tilde{\phi}_p)}((\boldsymbol{y}_{t-1} - \mu_1\mathbf{1}_p) \otimes (\boldsymbol{y}_{t-1} - \mu_1\mathbf{1}_p)), \\
d_3(\boldsymbol{y}_{t-1};\tilde{\phi}) &= -\frac{p}{2\tilde{\sigma}_1^2} + \frac{1}{2\tilde{\sigma}_1^2}(\boldsymbol{y}_{t-1} - \mu_1\mathbf{1}_p)'\boldsymbol{\Gamma}_{1,p}^{-1}(\boldsymbol{y}_{t-1} - \mu_1\mathbf{1}_p)
\end{aligned}$$

(first and last scalars, middle one $p \times 1$). Therefore

$$\frac{\nabla \mathsf{n}_p(\pi^*)}{\mathsf{n}_p(\pi^*)} = \nabla \log(\mathsf{n}_p(\boldsymbol{y}_{t-1};\pi^*)) = \begin{bmatrix} d_1(\boldsymbol{y}_{t-1};\pi^*) \\ d_2(\boldsymbol{y}_{t-1};\pi^*) \\ d_3(\boldsymbol{y}_{t-1};\pi^*) \end{bmatrix}.$$

Based on the preceding derivations, the derivatives appearing in (40)–(42) can now be expressed as

$$\begin{aligned}
\nabla_\beta l_t^\pi(\alpha,\beta^*,\pi^*,0) &= \frac{1}{\tilde{\sigma}_1^*}\varepsilon_t \\
\nabla_\pi l_t^\pi(\alpha,\beta^*,\pi^*,0) &= \begin{bmatrix} \frac{1}{\tilde{\sigma}_1^*}\boldsymbol{y}_{t-1}\varepsilon_t \\ \frac{1}{2\tilde{\sigma}_1^{*2}}(\varepsilon_t^2 - 1) \end{bmatrix} \qquad ((p+1)\times 1) \\
\nabla^2_{\varpi\varpi'} l_t^\pi(\alpha,\beta^*,\pi^*,0) &= \alpha(1-\alpha)\left\{ \begin{bmatrix} d_2(\boldsymbol{y}_{t-1};\pi^*)\frac{1}{\tilde{\sigma}_1^*}\boldsymbol{y}_{t-1}'\varepsilon_t & d_2(\boldsymbol{y}_{t-1};\pi^*)\frac{1}{2\tilde{\sigma}_1^{*2}}(\varepsilon_t^2 - 1) \\ d_3(\boldsymbol{y}_{t-1};\pi^*)\frac{1}{\tilde{\sigma}_1^*}\boldsymbol{y}_{t-1}'\varepsilon_t & d_3(\boldsymbol{y}_{t-1};\pi^*)\frac{1}{2\tilde{\sigma}_1^{*2}}(\varepsilon_t^2 - 1) \end{bmatrix} \right. \\
&\quad + \begin{bmatrix} \frac{1}{\tilde{\sigma}_1^*}\boldsymbol{y}_{t-1}\varepsilon_t d_2'(\boldsymbol{y}_{t-1};\pi^*) & \frac{1}{\tilde{\sigma}_1^*}\boldsymbol{y}_{t-1}\varepsilon_t d_3(\boldsymbol{y}_{t-1};\pi^*) \\ \frac{1}{2\tilde{\sigma}_1^{*2}}(\varepsilon_t^2 - 1)d_2'(\boldsymbol{y}_{t-1};\pi^*) & \frac{1}{2\tilde{\sigma}_1^{*2}}(\varepsilon_t^2 - 1)d_3(\boldsymbol{y}_{t-1};\pi^*) \end{bmatrix} \\
&\quad \left. + \begin{bmatrix} \frac{1}{\tilde{\sigma}_1^{*2}}\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}'(\varepsilon_t^2 - 1) & \frac{1}{2\tilde{\sigma}_1^{*3}}\boldsymbol{y}_{t-1}(\varepsilon_t^3 - 3\varepsilon_t) \\ \frac{1}{2\tilde{\sigma}_1^{*3}}\boldsymbol{y}_{t-1}'(\varepsilon_t^3 - 3\varepsilon_t) & \frac{1}{4\tilde{\sigma}_1^{*4}}(\varepsilon_t^4 - 6\varepsilon_t^2 + 3) \end{bmatrix} \right\}.
\end{aligned}$$

Verification of Assumption 5(iii). Finiteness of $\mathcal{I}$ was already established in Supplementary Appendix F.2. For positive definiteness, it suffices to show that the components of the vector s_t are linearly independent. Note that for linear independence, it does not matter if the order of the elements is changed or if some of the elements are multiplied by nonzero constants. Therefore, making use of the explicit expressions given above, it suffices to show that the components of the vector $\tilde{s}_t = (\tilde{s}_{t,1}, \tilde{s}_{t,2}, \tilde{s}_{t,3}, \tilde{s}_{t,4}, \tilde{s}_{t,5}, \tilde{s}_{t,6})$ (where the dimensions of the six components are $1, p, 1, p(p+1)/2, p, 1,$

respectively) are linearly independent, where

$$\begin{bmatrix}\tilde{s}_{t,1}\\ \tilde{s}_{t,2}\\ \tilde{s}_{t,3}\\ \tilde{s}_{t,4}\\ \tilde{s}_{t,5}\\ \tilde{s}_{t,6}\end{bmatrix} = \begin{bmatrix} \frac{1}{\tilde{\sigma}_1^*}\varepsilon_t \\ \frac{1}{\tilde{\sigma}_1^*}\boldsymbol{y}_{t-1}\varepsilon_t \\ \frac{1}{2\tilde{\sigma}_1^{*2}}(\varepsilon_t^2-1) \\ vech[d_2(\boldsymbol{y}_{t-1};\pi^*)\boldsymbol{y}_{t-1}' + \boldsymbol{y}_{t-1}d_2'(\boldsymbol{y}_{t-1};\pi^*)]\frac{1}{\tilde{\sigma}_1^*}\varepsilon_t + vech[\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}']\frac{1}{\tilde{\sigma}_1^{*2}}(\varepsilon_t^2-1) \\ d_2(\boldsymbol{y}_{t-1};\pi^*)\frac{1}{2\tilde{\sigma}_1^{*2}}(\varepsilon_t^2-1) + \boldsymbol{y}_{t-1}d_3(\boldsymbol{y}_{t-1};\pi^*)\frac{1}{\tilde{\sigma}_1^*}\varepsilon_t + \frac{1}{2\tilde{\sigma}_1^{*3}}\boldsymbol{y}_{t-1}(\varepsilon_t^3-3\varepsilon_t) \\ d_3(\boldsymbol{y}_{t-1};\pi^*)\frac{1}{\tilde{\sigma}_1^{*2}}(\varepsilon_t^2-1) + \frac{1}{4\tilde{\sigma}_1^{*4}}(\varepsilon_t^4-6\varepsilon_t^2+3)\end{bmatrix}.$$

To this end, suppose that $c'\tilde{s}_t = (c_1,c_2,c_3,c_4,c_5,c_6)'(\tilde{s}_{t,1},\tilde{s}_{t,2},\tilde{s}_{t,3},\tilde{s}_{t,4},\tilde{s}_{t,5},\tilde{s}_{t,6}) = 0$ (with the dimension of c and its subvectors chosen conformably). Note that the only random quantities $\tilde{s}_t$ depends on are $\boldsymbol{y}_{t-1}$ and ε_t which are independent. First, as the term ε_t^4 only appears in $\tilde{s}_{t,6}$, the equality $E[c'\tilde{s}_t \mid \varepsilon_t] = 0$ can be expressed as $c_6\varepsilon_t^4/(4\tilde{\sigma}_1^{*4}) + P_3(\varepsilon_t) = 0$, where $P_3(\varepsilon_t)$ is a third-order polynomial in ε_t. As the components of the vector $(\varepsilon_t,\varepsilon_t^2,\varepsilon_t^3,\varepsilon_t^4)$ are linearly independent (this clearly follows from normality), it follows that $c_6 = 0$. Next, basic properties of the standard normal distribution imply that $E[c'\tilde{s}_t(\varepsilon_t^3-3\varepsilon_t) \mid \boldsymbol{y}_{t-1}] = c_5'\frac{1}{2\tilde{\sigma}_1^{*3}}\boldsymbol{y}_{t-1}E[(\varepsilon_t^3-3\varepsilon_t)^2] = 0$, so that necessarily $c_5 = 0$ (as the components of $\boldsymbol{y}_{t-1}$ are linearly independent and $E[(\varepsilon_t^3-3\varepsilon_t)^2] > 0$). Next note that (as $c_5 = 0$, $c_6 = 0$)

$$0 = E[c'\tilde{s}_t(\varepsilon_t^2-1) \mid \boldsymbol{y}_{t-1}] = c_3E[(\varepsilon_t^2-1)^2]/(2\tilde{\sigma}_1^{*2}) + c_4'vech[\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}']E[(\varepsilon_t^2-1)^2]/\tilde{\sigma}_1^{*2}$$

so that $c_4'vech[\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}'] = -c_3/2$. As the components of $vech[\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}']$ are linearly independent (as $vech(\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}') = D_p^+vec(\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}') = D_p^+(\boldsymbol{y}_{t-1}\otimes\boldsymbol{y}_{t-1})$, with D_p^+ denoting the Moore-Penrose inverse of the duplication matrix D_p, $Cov[vech(\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}')] = D_p^+Cov(\boldsymbol{y}_{t-1}\otimes\boldsymbol{y}_{t-1})D_p^{+\prime}$; because D_p^+ is of full row rank and $Cov(\boldsymbol{y}_{t-1}\otimes\boldsymbol{y}_{t-1})$ has rank $p(p+1)/2$, see Thm 4.3(v) of Magnus and Neudecker (1979), $Cov[vech(\boldsymbol{y}_{t-1}\boldsymbol{y}_{t-1}')]$ is positive definite), it necessarily follows that $c_4 = 0$ and $c_3 = 0$. Finally, as only c_1 and c_2 may be nonzero, $E[c'\tilde{s}_t\varepsilon_t \mid \boldsymbol{y}_{t-1}] = c_1\frac{1}{\tilde{\sigma}_1^*} + c_2'\frac{1}{\tilde{\sigma}_1^*}\boldsymbol{y}_{t-1} = 0$, from which $c_2 = 0$ and $c_1 = 0$ follow (as the components of $\boldsymbol{y}_{t-1}$ are linearly independent).

F.4 Verification of Assumption 5(iv)

First consider $R_T^{(1)}(\alpha,\tilde{\pi},\varpi)$. Of the quantities on the right hand side of (48), the first two are equal to zero because $\nabla_{\varpi}l_t^{\pi}(\alpha,\tilde{\pi}^*,0) = 0$ and $\nabla_{\tilde{\pi}\varpi'}l_t^{\pi}(\alpha,\tilde{\pi}^*,0) = 0$; for the other eight quantities, Lemma F.4 provides upper bounds that aid in bounding them. Now, to verify Assumption 5(iv) (for $R_T^{(1)}(\alpha,\tilde{\pi},\varpi)$), let $\{\gamma_T,\ T\geq 1\}$ be an arbitrary sequence of (non-random) positive scalars such that $\gamma_T \to 0$ as $T\to\infty$. Condition $\|(\beta,\pi,\varpi)-(\beta^*,\pi^*,0)\| \leq \gamma_T$ (appearing in Assumption 5(iv)), together with the properties $\|\boldsymbol{\theta}(\alpha,\beta,\pi,\varpi)\|^2 = \|\tilde{\pi}-\tilde{\pi}^*\|^2 + \alpha_1^2\alpha_2^2\|v(\varpi)\|^2$ and $\|v(\varpi)\| \leq \|\varpi\|^2$, implies that

$$\|\boldsymbol{\theta}(\alpha,\beta,\pi,\varpi)\|^{1/2} \leq \|\tilde{\pi}-\tilde{\pi}^*\|^{1/2} + \alpha_1^{1/2}\alpha_2^{1/2}\|v(\varpi)\|^{1/2} \leq \gamma_T^{1/2} + \alpha_1^{1/2}\alpha_2^{1/2}\gamma_T.$$

This, Lemma F.4, and the fact that α is bounded away from zero and one on A, imply that for some sequence $\{\tilde{\gamma}_T,\ T\geq 1\}$ of (non-random) positive scalars such that $\tilde{\gamma}_T \to 0$ as $T\to\infty$ and for some finite C,

$$\sup_{\alpha\in A}\sup_{\substack{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha,\\ \|(\tilde{\pi},\varpi)-(\tilde{\pi}^*,0)\|\leq\gamma_T}}\frac{|R_T^{(1)}(\alpha,\tilde{\pi},\varpi)|}{\left(1+\|T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\|\right)^2}$$
$$\leq \tilde{\gamma}_T\sum_{i,j,k}\sup_{\alpha\in A}\left[|T^{-1}\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k}L_T^\pi(\alpha,\tilde{\pi}^*,0)|+|T^{-1}\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\varpi_k}L_T^\pi(\alpha,\tilde{\pi}^*,0)|+|T^{-1/2}\nabla^3_{\varpi_i\varpi_j\varpi_k}L_T^\pi(\alpha,\tilde{\pi}^*,0)|\right]$$
$$+\tilde{\gamma}_T\sum_{i,j,k,l}\sup_{\alpha\in A}\sup_{\substack{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha,\\ \|(\tilde{\pi},\varpi)-(\tilde{\pi}^*,0)\|\leq\gamma_T}}\left[|T^{-1}\nabla^4_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k\tilde{\pi}_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)|+|T^{-1}\nabla^4_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)|\right.$$
$$\left.+|T^{-1}\nabla^4_{\tilde{\pi}_i\tilde{\pi}_j\varpi_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)|+|T^{-1}\nabla^4_{\tilde{\pi}_i\varpi_j\varpi_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)|\right]$$
$$+C\sum_{i,j,k,l}\sup_{\alpha\in A}\sup_{\substack{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha,\\ \|(\tilde{\pi},\varpi)-(\tilde{\pi}^*,0)\|\leq\gamma_T}}|T^{-1}(\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)-\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi}^*,0))|, \tag{54}$$

where the summations above are understood to contain counterparts of each term in (48). As the data is assumed to be generated by a linear autoregression (Assumption 1), the y_t's form a stationary and ergodic process. Moreover, as the reparameterized log-likelihood of the GMAR model is four times continuously differentiable (see Assumption 4 and its verification), also the $\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k}l_t^\pi(\alpha,\tilde{\pi}^*,0)$'s form a stationary and ergodic process (for any i,j,k). An analogous result holds for all the third and fourth partial derivatives of $l_t^\pi(\alpha,\tilde{\pi},\varpi)$ appearing on the majorant side of (54).

Now, Lemma F.2(iii) together with the ergodic theorem implies that $\sup_{\alpha\in A}[|T^{-1}\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k}L_T^\pi(\alpha,\tilde{\pi}^*,0)|]=O_p(1)$ (for any i,j,k). Similarly, $\sup_{\alpha\in A}[|T^{-1}\nabla^3_{\tilde{\pi}_i\tilde{\pi}_j\varpi_k}L_T^\pi(\alpha,\tilde{\pi}^*,0)|]=O_p(1)$ (for any i,j,k). Expression of $\nabla^3_{\varpi_i\varpi_j\varpi_k}l_t^\pi(\alpha,\tilde{\pi}^*,0)$ in Supplementary Appendix F.1, Lemmas F.1 and F.3, and the compactness of A, imply that $\sup_{\alpha\in A}|T^{-1/2}\nabla^3_{\varpi_i\varpi_j\varpi_k}L_T^\pi(\alpha,\tilde{\pi}^*,0)|\leq C|T^{-1/2}\sum_{t=1}^T MDS_{t,i,j,k}(\pi^*)|$ for some finite C and for some square integrable martingale difference sequence $MDS_{t,i,j,k}(\pi^*)$. Moreover, for any i,j,k, the last upper bound is $O_p(1)$ by an appropriate central limit theorem (Billingsley (1961)).

As for the fourth-order partial derivatives appearing on the majorant side of (54), Lemma F.2(iv) and a uniform law of large numbers for stationary and ergodic processes (Ranga Rao (1962)) imply that $\sup_{\alpha\in A}\sup_{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha,\|(\tilde{\pi},\varpi)-(\tilde{\pi}^*,0)\|\leq\gamma_T}|T^{-1}\nabla^4_{\tilde{\pi}_i\tilde{\pi}_j\tilde{\pi}_k\tilde{\pi}_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)|=O_p(1)$ (for any i,j,k,l). The next three terms in (54) can be handled similarly. As for the last term on the majorant side of (54),

$$\sup_{\alpha\in A}\sup_{\substack{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha,\\ \|(\tilde{\pi},\varpi)-(\tilde{\pi}^*,0)\|\leq\gamma_T}}|T^{-1}(\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)-\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi}^*,0))|$$
$$\leq 2\sup_{\alpha\in A}\sup_{\substack{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha,\\ \|(\tilde{\pi},\varpi)-(\tilde{\pi}^*,0)\|\leq\gamma_T}}|T^{-1}\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^\pi(\alpha,\tilde{\pi},\varpi)-E[\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}l_t^\pi(\alpha,\tilde{\pi},\varpi)]|$$

where the dominant side is $o_p(1)$ (again relying on Lemma F.2(iv) and a uniform LLN). To conclude, the upper bound in (54) is $\tilde{\gamma}_TO_p(1)+Co_p(1)=o_p(1)$. This completes the verification of Assumption 5(iv) for the term $R_T^{(1)}(\alpha,\tilde{\pi},\varpi)$.

Now consider $R_T^{(2)}(\alpha,\tilde{\pi},\varpi)=-\frac{1}{2}[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]'(\mathcal{J}_T-\mathcal{I})[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]$. We will below show that (a) $\mathcal{J}_T\xrightarrow{p}\mathcal{J}$ as $T\to\infty$, where the matrix $\mathcal{J}$ will be specified below (and $\mathcal{J}_T,\mathcal{J}$ do not depend on α). Write (-2 times) $R_T^{(2)}(\alpha,\tilde{\pi},\varpi)$ as

$$[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]'(\mathcal{J}_T-\mathcal{J})[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]+[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)]'(\mathcal{J}-\mathcal{I})[T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)].$$

We will below also show that (b) the latter term above equals zero. The validity of Assumption 5(iv) for the term $R_T^{(2)}(\alpha,\tilde{\pi},\varpi)$ follows from results (a) and (b) (together with usual properties of the Euclidean norm).

To prove claim (a), we first define the matrix $\mathcal{J}$ as

$$\mathcal{J} = \begin{bmatrix} \mathcal{J}_{\pi\pi} & \mathcal{J}'_{\pi\varpi\varpi} \\ \mathcal{J}_{\pi\varpi\varpi} & \mathcal{J}_{\varpi\varpi\varpi\varpi} \end{bmatrix}$$

where the matrices $\mathcal{J}_{\tilde{\pi}\tilde{\pi}}$ $((q_1+q_2)\times(q_1+q_2))$, $\mathcal{J}'_{\tilde{\pi}\varpi\varpi}$ $((q_1+q_2)\times q_\vartheta)$, and $\mathcal{J}_{\varpi\varpi\varpi\varpi}$ $(q_\vartheta \times q_\vartheta)$ are defined as

$$\mathcal{J}_{\tilde{\pi}\tilde{\pi}} = E\left[\frac{\nabla f_t^*}{f_t^*}\frac{\nabla' f_t^*}{f_t^*}\right]$$

$$\mathcal{J}'_{\tilde{\pi}\varpi\varpi} = E\left[\left[c_{ij}\frac{\nabla f_t^*}{f_t^*}X^*_{t,i,j}\right]_{(i,j)\in\mathfrak{I}}\right]$$

$$\mathcal{J}_{\varpi\varpi\varpi\varpi} = \frac{1}{3}\left[c_{ij}c_{kl}\left(E[X^*_{t,i,j}X^*_{t,k,l}] + E[X^*_{t,i,k}X^*_{t,j,l}] + E[X^*_{t,i,l}X^*_{t,j,k}]\right)\right]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}}$$

where the $X^*_{t,i,j}$ $(i,j\in\{1,\ldots,q_2\})$ are as in (44). Finiteness of $\mathcal{J}$ follows from Lemma F.1.

Now consider the convergence result $\mathcal{J}_T \xrightarrow{p} \mathcal{J}$ for each block at a time. For the top-left block, from Supplementary Appendix F.1 we have $\nabla_{\tilde{\pi}\tilde{\pi}'} l_t^{\pi*} = \frac{\nabla^2 f_t^*}{f_t^*} - \frac{\nabla f_t^*}{f_t^*}\frac{\nabla' f_t^*}{f_t^*}$ so that ergodic theorem and Lemmas F.1 and F.3 (latter ensuring the first term on the right-hand side of the previous equation has zero expectation) imply that $\mathcal{J}_{T,\tilde{\pi}\tilde{\pi}} = -T^{-1}\nabla^2_{\tilde{\pi}\tilde{\pi}'}L_T^{\pi*} \xrightarrow{p} \mathcal{J}_{\tilde{\pi}\tilde{\pi}}$.

For the off-diagonal block, consider the expression of $\nabla^3_{\tilde{\pi}_i\varpi_j\varpi_k} l_t^{\pi*}$ in Supplementary Appendix F.1. Lemma F.3 ensures that of the ten summands in this expression, only the second, fourth, and sixth ones have non-zero expectation. Therefore the ergodic theorem and Lemma F.1 imply that

$$\mathcal{J}'_{T,\tilde{\pi}\varpi\varpi} = -T^{-1}\frac{1}{\alpha_1\alpha_2}[c_{11}\nabla^3_{\tilde{\pi}\varpi_1\varpi_1}L_T^{\pi*} : \cdots : c_{q_2-1,q_2}\nabla^3_{\tilde{\pi}\varpi_{q_2-1}\varpi_{q_2}}L_T^{\pi*}] \xrightarrow{p} \mathcal{J}'_{\tilde{\pi}\varpi\varpi}.$$

Lastly, for the bottom-right block, consider the expression of $\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l} l_t^{\pi*}$ in Supplementary Appendix F.1. Lemma F.3 reveals that the terms in this expression that have non-zero expectation can be expressed as

$$-\alpha_1^2\alpha_2^2[X^*_{t,i,j}X^*_{t,k,l} + X^*_{t,i,k}X^*_{t,j,l} + X^*_{t,i,l}X^*_{t,j,k}].$$

Therefore the ergodic theorem and Lemma F.1 imply that

$$\mathcal{J}_{T,\varpi\varpi\varpi\varpi} = -T^{-1}\frac{8}{4!}\frac{1}{\alpha_1^2\alpha_2^2}\left[c_{ij}c_{kl}\nabla^4_{\varpi_i\varpi_j\varpi_k\varpi_l}L_T^{\pi*}\right]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}} \xrightarrow{p} \mathcal{J}_{\varpi\varpi\varpi\varpi}.$$

This completes the proof of claim (a).

To prove claim (b), first note that from the definitions of $\mathcal{J}$ and $\mathcal{I}$ (see (51)) it can be seen that only the bottom-right blocks of $\mathcal{J}$ and $\mathcal{I}$ differ. Therefore, as $T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi) = (T^{1/2}(\tilde{\pi} - \tilde{\pi}^*), T^{1/2}(\alpha_1\alpha_2 v(\varpi)))$, claim (b) holds if $T(\alpha_1\alpha_2)^2 v(\varpi)'(\mathcal{J}_{\varpi\varpi\varpi\varpi} - \mathcal{I}_{\varpi\varpi\varpi\varpi})v(\varpi) = 0$ where

$$\mathcal{J}_{\varpi\varpi\varpi\varpi} = \frac{1}{3}\left[c_{ij}c_{kl}\left(E[X^*_{t,i,j}X^*_{t,k,l}] + E[X^*_{t,i,k}X^*_{t,j,l}] + E[X^*_{t,i,l}X^*_{t,j,k}]\right)\right]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}},$$

$$\mathcal{I}_{\varpi\varpi\varpi\varpi} = \left[c_{ij}c_{kl}E[X^*_{t,i,j}X^*_{t,k,l}]\right]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}}.$$

Note that the scalars $A_{ijkl} = E[X^*_{t,i,j}X^*_{t,k,l}]$ satisfy $A_{ijkl} = A_{jikl}$ and $A_{ijkl} = A_{ijlk}$ for all i,j,k,l so that using property (49) we obtain

$$\begin{aligned}
v(\varpi)'\mathcal{J}_{\varpi\varpi\varpi\varpi}v(\varpi) &= \frac{1}{3}v(\varpi)'\left[c_{ij}c_{kl}\left(E[X^*_{t,i,j}X^*_{t,k,l}]+E[X^*_{t,i,k}X^*_{t,j,l}]+E[X^*_{t,i,l}X^*_{t,j,k}]\right)\right]_{(i,j,k,l)\in\mathfrak{I}\times\mathfrak{I}}v(\varpi) \\
&= \frac{1}{3}\frac{1}{4}\sum_{i=1}^{q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2}\left(E[X^*_{t,i,j}X^*_{t,k,l}]+E[X^*_{t,i,k}X^*_{t,j,l}]+E[X^*_{t,i,l}X^*_{t,j,k}]\right)\varpi_i\varpi_j\varpi_k\varpi_l \\
&= \frac{1}{4}\sum_{i=1}^{q_2}\sum_{j=1}^{q_2}\sum_{k=1}^{q_2}\sum_{l=1}^{q_2}E[X^*_{t,i,j}X^*_{t,k,l}]\varpi_i\varpi_j\varpi_k\varpi_l \\
&= v(\varpi)'\mathcal{I}_{\varpi\varpi\varpi\varpi}v(\varpi).
\end{aligned}$$

This completes the proof of claim (b).

Therefore, the verification of Assumption 5(iv) is done.

F.5 Additional Lemmas

The following four lemmas contain results needed in the proofs. Note that the first and the third lemma are not specific to the examples in this paper, whereas the second and fourth lemmas concern only the GMAR example. In the first lemma, $\mathsf{n}_{p+1}(\tilde{\phi}) = \mathsf{n}_{p+1}(y_t, \boldsymbol{y}_{t-1}; \tilde{\phi})$ denotes the $(p+1)$-dimensional density function of an AR(p) process based on parameter value $\tilde{\phi}$ evaluated at $(y_t, \boldsymbol{y}_{t-1})$; cf. equations (6)–(8) for the p-dimensional counterpart $\mathsf{n}_p(\tilde{\phi}) = \mathsf{n}_p(\boldsymbol{y}_{t-1}; \tilde{\phi})$.

Lemma F.1. *For any $i,j,k,l \in \{1,\ldots,p+2\}$ and any positive r, the following moments are all finite:*

(i) $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla_i f_t(\tilde{\phi})/f_t(\tilde{\phi})|^r\big]$, $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla^2_{ij} f_t(\tilde{\phi})/f_t(\tilde{\phi})|^r\big], \ldots, E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla^4_{ijkl} f_t(\tilde{\phi})/f_t(\tilde{\phi})|^r\big]$,

(ii) $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla_i \mathsf{n}_p(\tilde{\phi})/\mathsf{n}_p(\tilde{\phi})|^r\big]$, $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla^2_{ij} \mathsf{n}_p(\tilde{\phi})/\mathsf{n}_p(\tilde{\phi})|^r\big], \ldots, E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla^4_{ijkl} \mathsf{n}_p(\tilde{\phi})/\mathsf{n}_p(\tilde{\phi})|^r\big]$,

(iii) $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla_i \mathsf{n}_{p+1}(\tilde{\phi})|^r\big]$, $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla^2_{ij} \mathsf{n}_{p+1}(\tilde{\phi})|^r\big], \ldots, E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}|\nabla^4_{ijkl} \mathsf{n}_{p+1}(\tilde{\phi})|^r\big]$.

Lemma F.2. *In the GMAR example the following hold, where each of (the scalars) z_1, z_2, z_3, z_4 is a 'placeholder' for any of $\tilde{\pi}_i, \tilde{\pi}_j, \tilde{\pi}_k, \tilde{\pi}_l$ $(i,j,k,l \in \{1,\ldots,q_1+q_2\})$ or $\varpi_i, \varpi_j, \varpi_k, \varpi_l$ $(i,j,k,l \in \{1,\ldots,q_2\})$:*

(i) $E\left[\sup_{\alpha\in A}\sup_{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha}|\nabla_{z_1} l^\pi_t(\alpha,\tilde{\pi},\varpi)|\right] < \infty$,

(ii) $E\left[\sup_{\alpha\in A}\sup_{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha}|\nabla^2_{z_1z_2} l^\pi_t(\alpha,\tilde{\pi},\varpi)|\right] < \infty$,

(iii) $E\left[\sup_{\alpha\in A}\sup_{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha}|\nabla^3_{z_1z_2z_3} l^\pi_t(\alpha,\tilde{\pi},\varpi)|\right] < \infty$,

(iv) $E\left[\sup_{\alpha\in A}\sup_{(\tilde{\pi},\varpi)\in B\times\Pi_\alpha}|\nabla^4_{z_1z_2z_3z_4} l^\pi_t(\alpha,\tilde{\pi},\varpi)|\right] < \infty$.

Lemma F.3. *For any $i,j,k,l \in \{1,\ldots,p+2\}$,*

$$E\Big[\ \frac{\nabla_i f^*_t}{f^*_t}\ \mid \boldsymbol{y}_{t-1}\Big] = E\Big[\ \frac{\nabla^2_{ij} f^*_t}{f^*_t}\ \mid \boldsymbol{y}_{t-1}\Big] = E\Big[\ \frac{\nabla^3_{ijk} f^*_t}{f^*_t}\ \mid \boldsymbol{y}_{t-1}\Big] = E\Big[\ \frac{\nabla^4_{ijkl} f^*_t}{f^*_t}\ \mid \boldsymbol{y}_{t-1}\Big] = 0.$$

Lemma F.4. *In the GMAR example the following hold for all $\alpha \in A$, $(\tilde{\pi}, \varpi) \in B \times \Pi_\alpha$, T, and $i, j, k, l \in \{1, \ldots, q_1 + q_2\}$ (subindex in $\tilde{\pi}$) or $i, j, k, l \in \{1, \ldots, q_2\}$ (subindex in ϖ):*

$$\text{(i)} \quad T\,|\tilde{\pi}_i - \tilde{\pi}_i^*|\,|\tilde{\pi}_j - \tilde{\pi}_j^*||\tilde{\pi}_k - \tilde{\pi}_k^*| \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 \|\tilde{\pi} - \tilde{\pi}^*\|,$$

$$\text{(ii)} \quad T|\tilde{\pi}_i - \tilde{\pi}_i^*||\tilde{\pi}_j - \tilde{\pi}_j^*||\varpi_k| \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 \|\varpi\|,$$

$$\text{(iii)} \quad T^{1/2}|\varpi_i||\varpi_j||\varpi_k| \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 (\alpha_1\alpha_2)^{-3/2} \|\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|^{1/2},$$

$$\text{(iv)} \quad T(\tilde{\pi}_i - \tilde{\pi}_i^*)(\tilde{\pi}_j - \tilde{\pi}_j^*)(\tilde{\pi}_k - \tilde{\pi}_k^*)(\tilde{\pi}_l - \tilde{\pi}_l^*) \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 \|\tilde{\pi} - \tilde{\pi}^*\|^2,$$

$$\text{(v)} \quad T(\tilde{\pi}_i - \tilde{\pi}_i^*)(\tilde{\pi}_j - \tilde{\pi}_j^*)(\tilde{\pi}_k - \tilde{\pi}_k^*)\varpi_l \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 \|\tilde{\pi} - \tilde{\pi}^*\|\,\|\varpi\|,$$

$$\text{(vi)} \quad T(\tilde{\pi}_i - \tilde{\pi}_i^*)(\tilde{\pi}_j - \tilde{\pi}_j^*)\varpi_k\varpi_l \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 \|\varpi\|^2,$$

$$\text{(vii)} \quad T(\tilde{\pi}_i - \tilde{\pi}_i^*)\varpi_j\varpi_k\varpi_l \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 (\alpha_1\alpha_2)^{-3/2} \|\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|^{1/2},$$

$$\text{(viii)} \quad T\varpi_i\varpi_j\varpi_k\varpi_l \leq (1 + \|T^{1/2}\boldsymbol{\theta}(\alpha, \tilde{\pi}, \varpi)\|)^2 (\alpha_1\alpha_2)^{-2}.$$

F.6 Proofs of Lemmas F.1–F.4

Proof of Lemma F.1. Writing $g_t(\tilde{\phi}) = [y_t - (\tilde{\phi}_0 + \tilde{\phi}_1 y_{t-1} + \cdots + \tilde{\phi}_p y_{t-p})]/\tilde{\sigma}_1$ and recalling the definition of $f_t(\tilde{\phi})$ we can write $f_t(\tilde{\phi}) = \tilde{\sigma}_1^{-1}\mathfrak{n}(g_t(\tilde{\phi}))$ where $\mathfrak{n}(\cdot)$ denotes the density function of a standard normal random variable. Recall also that derivatives of $\mathfrak{n}(\cdot)$ can be expressed using (one version of) Hermite polynomials $H_n(x)$ as $\frac{d^n}{dx^n}\mathfrak{n}(x) = (-1)^n H_n(x)\mathfrak{n}(x)$. Using the chain rule for differentiation repeatedly, it can therefore be seen that each of the functions $\nabla_i f_t(\tilde{\phi})/f_t(\tilde{\phi})$, $\nabla_{ij}^2 f_t(\tilde{\phi})/f_t(\tilde{\phi})$, $\nabla_{ijk}^3 f_t(\tilde{\phi})/f_t(\tilde{\phi})$, and $\nabla_{ijkl}^4 f_t(\tilde{\phi})/f_t(\tilde{\phi})$ can be expressed as a sum of terms each of which is a product involving Hermite polynomials $H_n(g_t(\tilde{\phi}))$ and powers of derivatives of $g_t(\tilde{\phi})$ (and functions of $\tilde{\phi}$). Thus, each of these functions is a polynomial in terms of $y_t, y_{t-1}, \ldots, y_{t-p}$. As the y_t's are generated by a stationary linear Gaussian AR(p) model, they possess moments of all orders, implying (together with the definition of $\tilde{\Phi}$, implying in particular that $\tilde{\sigma}_1$ is bounded away from zero on $\tilde{\Phi}$) the finiteness of the moments listed in part (i) of the lemma.

As for part (ii), note that $\mathsf{n}_p(\tilde{\phi})$ can be expressed as $\mathsf{n}_p(\tilde{\phi}) = g_1(\tilde{\phi})\mathfrak{n}(g_{2,t}(\tilde{\phi}))$ for some function $g_1(\tilde{\phi})$ not depending on the y_t's and $g_{2,t}(\tilde{\phi})$ the square root of a second-order polynomial in $y_{t-1}, \ldots, y_{t-p}$. Therefore the finiteness of the moments listed in the part (ii) follows using similar arguments as above (noting that the definition of $\tilde{\Phi}$ implies that the determinant of the covariance matrix appearing in $\mathsf{n}_p(\tilde{\phi})$ is bounded away from zero on $\tilde{\Phi}$).

Finally, for part (iii), similar arguments, together with the observation that $\mathsf{n}_{p+1}(\tilde{\phi})$ is bounded on $\tilde{\Phi}$, yield the desired result. ■

Proof of Lemma F.2. To prove (i), first consider the derivatives with respect to ϖ. From the formulas in Supplementary Appendix F.7 we obtain

$$\begin{aligned}
\nabla_\varpi l_t^\pi(\alpha, \tilde{\pi}, \varpi) = {} & \alpha_{1,t}\alpha_{2,t}\left(\frac{D_{\tilde{\phi},\varpi}^{(1)\prime}\nabla\mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} - \frac{D_{\tilde{\varphi},\varpi}^{(1)\prime}\nabla\mathsf{n}_p(\tilde{\varphi})}{\mathsf{n}_p(\tilde{\varphi})}\right)\frac{f_t(\tilde{\phi})}{f_{2,t}^\pi(\alpha, \tilde{\pi}, \varpi)} + \frac{D_{\tilde{\phi},\varpi}^{(1)\prime}\nabla f_t(\tilde{\phi})}{f_{2,t}^\pi(\alpha, \tilde{\pi}, \varpi)}\alpha_{1,t} \\
& - \alpha_{1,t}\alpha_{2,t}\left(\frac{D_{\tilde{\phi},\varpi}^{(1)\prime}\nabla\mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} - \frac{D_{\tilde{\varphi},\varpi}^{(1)\prime}\nabla\mathsf{n}_p(\tilde{\varphi})}{\mathsf{n}_p(\tilde{\varphi})}\right)\frac{f_t(\tilde{\varphi})}{f_{2,t}^\pi(\alpha, \tilde{\pi}, \varpi)} + \frac{D_{\tilde{\varphi},\varpi}^{(1)\prime}\nabla f_t(\tilde{\varphi})}{f_{2,t}^\pi(\alpha, \tilde{\pi}, \varpi)}(1 - \alpha_{1,t})
\end{aligned}$$

where $\tilde{\phi}$ and $\tilde{\varphi}$ are understood as functions of $(\alpha, \tilde{\pi}, \varpi)$ (i.e., $\tilde{\phi} = (\beta, \pi + \alpha_2\varpi)$ and $\tilde{\varphi} = (\beta, \pi - \alpha_1\varpi)$). Note that whenever $\alpha \in A$ and $(\tilde{\pi}, \varpi) \in B \times \Pi_\alpha$, $\tilde{\phi} \in \tilde{\Phi}$ and $\tilde{\varphi} \in \tilde{\Phi}$. Also note that over $\alpha \in A$ and

$(\tilde{\pi}, \varpi) \in B \times \Pi_\alpha$, the quantities

$$|\alpha_{1,t}|,\ |\alpha_{2,t}|,\ \|D^{(1)}_{\tilde{\phi},\varpi}\|,\ \|D^{(1)}_{\tilde{\varphi},\varpi}\|,\ |f_t(\tilde{\phi})/f^{\pi}_{2,t}(\alpha,\tilde{\pi},\varpi)|,\ |f_t(\tilde{\varphi})/f^{\pi}_{2,t}(\alpha,\tilde{\pi},\varpi)|$$

are all bounded by finite constants. Therefore $E\big[\sup_{\alpha\in A}\sup_{(\tilde{\pi},\varpi)\in N(\tilde{\pi}^*,0)}\|\nabla_\varpi l^\pi_t(\alpha,\tilde{\pi},\varpi)\|\big] < \infty$ as long as $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}\|\nabla f_t(\tilde{\phi})/f_t(\tilde{\phi})\|\big] < \infty$ and $E\big[\sup_{\tilde{\phi}\in\tilde{\Phi}}\|\nabla \mathsf{n}_p(\tilde{\phi})/\mathsf{n}_p(\tilde{\phi})\|\big] < \infty$, which is ensured by Lemma F.1. The argument for $\nabla_{\tilde{\pi}} l^\pi_t(\alpha,\tilde{\pi},\varpi)$ is entirely similar and is omitted.

To prove (ii)–(iv), entirely similar arguments can be used. Tedious calculations (details omitted) show that the finiteness of the required moments is ensured by the finiteness of the moments in Lemma F.1(i) and (ii). ■

Proof of Lemma F.3. For the first two derivatives, the stated result follows directly from the expressions of $\nabla f^*_t/f^*_t$ and $\nabla^2 f^*_t/f^*_t$ in (52). The results for the third and fourth derivatives can be obtained with straightforward calculation. ■

Proof of Lemma F.4. First recall that $\left\|T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\right\|^2 = T\,\|\tilde{\pi}-\tilde{\pi}^*\|^2 + T\alpha_1^2\alpha_2^2\,\|v(\varpi)\|^2$. (i) By an elementary inequality, $T\,|\tilde{\pi}_i-\tilde{\pi}^*_i|\,|\tilde{\pi}_j-\tilde{\pi}^*_j||\tilde{\pi}_k-\tilde{\pi}^*_k| \le T\,\|\tilde{\pi}-\tilde{\pi}^*\|^3$ and therefore the result follows by adding nonnegative terms on the majorant side of this inequality. Parts (ii) and (iv)–(vi) are shown similarly. (vii) As $\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi) = (\tilde{\pi}-\tilde{\pi}^*, \alpha_1\alpha_2 v(\varpi))$, each of the terms $|\tilde{\pi}_i-\tilde{\pi}^*_i|$, $\alpha_1\alpha_2\varpi_j^2$, $\alpha_1\alpha_2\varpi_k^2$, and $\alpha_1\alpha_2\varpi_l^2$ are dominated by $\|\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\|$. Therefore

$$T|\tilde{\pi}_i-\tilde{\pi}^*_i||\varpi_j||\varpi_k||\varpi_l| \le T(\alpha_1\alpha_2)^{-3/2}\,\|\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\|^{5/2} \le (\alpha_1\alpha_2)^{-3/2}(1+\left\|T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\right\|)^2\,\|\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\|^{1/2}$$

where the second inequality holds because nonnegative terms were added to the majorant side. (viii) Similarly as in the previous part,

$$T|\varpi_i||\varpi_j||\varpi_k||\varpi_l| \le (\alpha_1\alpha_2)^{-2}\left\|T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\right\|^2 \le (\alpha_1\alpha_2)^{-2}(1+\left\|T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\right\|)^2.$$

Finally, for (iii) we, similarly as above but scaling with $T^{1/2}$ instead of T, obtain

$$T^{1/2}|\varpi_i||\varpi_j||\varpi_k| \le T^{1/2}(\alpha_1\alpha_2)^{-3/2}\,\|\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\|^{3/2} \le (\alpha_1\alpha_2)^{-3/2}(1+\left\|T^{1/2}\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi\alpha)\right\|)^2\,\|\boldsymbol{\theta}(\alpha,\tilde{\pi},\varpi)\|^{1/2},$$

which completes the proof. ■

F.7 Partial derivatives of the reparameterized log-likelihood function (continued)

Note that $l^\pi_t(\alpha,\beta,\pi,\varpi) = \log[f^\pi_{2,t}(\alpha,\beta,\pi,\varpi)]$ with

$$\begin{aligned} f^\pi_{2,t}(\alpha,\beta,\pi,\varpi) &= \alpha^\pi_{1,t}(\alpha,\beta,\pi,\varpi)f_t(\beta,\pi+\alpha_2\varpi) + (1-\alpha^\pi_{1,t}(\alpha,\beta,\pi,\varpi))f_t(\beta,\pi-\alpha_1\varpi),\\ \alpha^\pi_{1,t}(\alpha,\beta,\pi,\varpi) &= \alpha^G_{1,t}(\alpha,(\beta,\pi+\alpha_2\varpi),(\beta,\pi-\alpha_1\varpi)). \end{aligned}$$

For the sake of brevity, but with slight abuse of notation, we will write these as

$$\begin{aligned} f^\pi_{2,t}(\alpha,\beta,\pi,\varpi) &= \alpha^\pi_{1,t}(\alpha,\beta,\pi,\varpi)f_t(\tilde{\phi}) + (1-\alpha^\pi_{1,t}(\alpha,\beta,\pi,\varpi))f_t(\tilde{\varphi}),\\ \alpha^\pi_{1,t}(\alpha,\beta,\pi,\varpi) &= \frac{\alpha\mathsf{n}_p(\tilde{\phi})}{\alpha\mathsf{n}_p(\tilde{\phi})+(1-\alpha)\mathsf{n}_p(\tilde{\varphi})}, \end{aligned}$$

where $\tilde{\phi}$ and $\tilde{\varphi}$ are understood as functions of $(\alpha, \beta, \pi, \varpi)$, that is, $\tilde{\phi} = (\beta, \pi + \alpha_2\varpi)$ and $\tilde{\varphi} = (\beta, \pi - \alpha_1\varpi)$.

The following notation will be helpful:

$$
\begin{aligned}
D^{(1)}_{\tilde{\phi},\tilde{\pi}} &= \frac{\partial(\beta, \pi + \alpha_2\varpi)}{\partial\tilde{\pi}'} = I_{1+q_2} \\
D^{(1)}_{\tilde{\phi},\varpi} &= \frac{\partial(\beta, \pi + \alpha_2\varpi)}{\partial\varpi'} = \begin{bmatrix} 0 \\ \alpha_2 I_{q_2} \end{bmatrix} \quad ((1+q_2)\times q_2) \\
D^{(1)}_{\tilde{\varphi},\tilde{\pi}} &= \frac{\partial(\beta, \pi - \alpha_1\varpi)}{\partial\tilde{\pi}'} = I_{1+q_2} \\
D^{(1)}_{\tilde{\varphi},\varpi} &= \frac{\partial(\beta, \pi - \alpha_1\varpi)}{\partial\varpi'} = \begin{bmatrix} 0 \\ -\alpha_1 I_{q_2} \end{bmatrix} \quad ((1+q_2)\times q_2)
\end{aligned}
$$

First-order partial derivatives. With straightforward differentation we obtain

$$
\begin{aligned}
\nabla_{\tilde{\pi}} l_t^{\pi}(\alpha, \tilde{\pi}, \varpi) &= \frac{\nabla_{\tilde{\pi}} f_{2,t}^{\pi}(\alpha, \tilde{\pi}, \varpi)}{f_{2,t}^{\pi}(\alpha, \tilde{\pi}, \varpi)} \\
\nabla_{\varpi} l_t^{\pi}(\alpha, \tilde{\pi}, \varpi) &= \frac{\nabla_{\varpi} f_{2,t}^{\pi}(\alpha, \tilde{\pi}, \varpi)}{f_{2,t}^{\pi}(\alpha, \tilde{\pi}, \varpi)}
\end{aligned}
$$

with

$$
\begin{aligned}
\nabla_{\tilde{\pi}} f_{2,t}^{\pi}(\alpha, \tilde{\pi}, \varpi) &= \nabla_{\tilde{\pi}}\alpha_{1,t} f_t(\tilde{\phi}) + \nabla f_t(\tilde{\phi})\alpha_{1,t} - \nabla_{\tilde{\pi}}\alpha_{1,t} f_t(\tilde{\varphi}) + \nabla f_t(\tilde{\varphi})(1-\alpha_{1,t}) \\
\nabla_{\varpi} f_{2,t}^{\pi}(\alpha, \tilde{\pi}, \varpi) &= \nabla_{\varpi}\alpha_{1,t} f_t(\tilde{\phi}) + D^{(1)\prime}_{\tilde{\phi},\varpi}\nabla f_t(\tilde{\phi})\alpha_{1,t} - \nabla_{\varpi}\alpha_{1,t} f_t(\tilde{\varphi}) + D^{(1)\prime}_{\tilde{\varphi},\varpi}\nabla f_t(\tilde{\varphi})(1-\alpha_{1,t})
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla_{\tilde{\pi}}\alpha_{1,t} &= \frac{\alpha_1 \nabla \mathsf{n}_p(\tilde{\phi})}{\alpha_1 \mathsf{n}_p(\tilde{\phi}) + \alpha_2 \mathsf{n}_p(\tilde{\varphi})} - \alpha_{1,t}\frac{\alpha_1 \nabla \mathsf{n}_p(\tilde{\phi}) + \alpha_2 \nabla \mathsf{n}_p(\tilde{\varphi})}{\alpha_1 \mathsf{n}_p(\tilde{\phi}) + \alpha_2 \mathsf{n}_p(\tilde{\varphi})} \\
\nabla_{\varpi}\alpha_{1,t} &= \frac{\alpha_1 D^{(1)\prime}_{\tilde{\phi},\varpi} \nabla \mathsf{n}_p(\tilde{\phi})}{\alpha_1 \mathsf{n}_p(\tilde{\phi}) + \alpha_2 \mathsf{n}_p(\tilde{\varphi})} - \alpha_{1,t}\frac{\alpha_1 D^{(1)\prime}_{\tilde{\phi},\varpi}\nabla \mathsf{n}_p(\tilde{\phi}) + \alpha_2 D^{(1)\prime}_{\tilde{\varphi},\varpi}\nabla \mathsf{n}_p(\tilde{\varphi})}{\alpha_1 \mathsf{n}_p(\tilde{\phi}) + \alpha_2 \mathsf{n}_p(\tilde{\varphi})}
\end{aligned}
$$

where simplification leads to

$$
\begin{aligned}
\nabla_{\tilde{\pi}}\alpha_{1,t} &= \alpha_{1,t}\frac{\nabla \mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} - \alpha_{1,t}\left(\alpha_{1,t}\frac{\nabla \mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} + \alpha_{2,t}\frac{\nabla \mathsf{n}_p(\tilde{\varphi})}{\mathsf{n}_p(\tilde{\varphi})}\right) \\
&= \alpha_{1,t}\alpha_{2,t}\left(\frac{\nabla \mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} - \frac{\nabla \mathsf{n}_p(\tilde{\varphi})}{\mathsf{n}_p(\tilde{\varphi})}\right) \\
\nabla_{\varpi}\alpha_{1,t} &= \alpha_{1,t}\frac{D^{(1)\prime}_{\tilde{\phi},\varpi}\nabla \mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} - \alpha_{1,t}\left(\alpha_{1,t}\frac{D^{(1)\prime}_{\tilde{\phi},\varpi}\nabla \mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} + \alpha_{2,t}\frac{D^{(1)\prime}_{\tilde{\varphi},\varpi}\nabla \mathsf{n}_p(\tilde{\varphi})}{\mathsf{n}_p(\tilde{\varphi})}\right) \\
&= \alpha_{1,t}\alpha_{2,t}\left(\frac{D^{(1)\prime}_{\tilde{\phi},\varpi}\nabla \mathsf{n}_p(\tilde{\phi})}{\mathsf{n}_p(\tilde{\phi})} - \frac{D^{(1)\prime}_{\tilde{\varphi},\varpi}\nabla \mathsf{n}_p(\tilde{\varphi})}{\mathsf{n}_p(\tilde{\varphi})}\right).
\end{aligned}
$$

Evaluated at $(\alpha, \tilde{\pi}, \varpi) = (\alpha, \tilde{\pi}^*, 0)$ we get

$$
\nabla_{\tilde{\pi}}\alpha_{1,t}^* = 0, \qquad \nabla_{\varpi}\alpha_{1,t}^* = \alpha_1\alpha_2\frac{\nabla_{(2,\ldots,p+2)}\mathsf{n}_p(\tilde{\pi}^*)}{\mathsf{n}_p(\tilde{\pi}^*)},
$$

$$f_{2,t}^{\pi}(\alpha,\tilde{\pi}^*,0) = f_t(\tilde{\pi}^*), \qquad \nabla_{\tilde{\pi}} f_{2,t}^{\pi}(\alpha,\tilde{\pi}^*,0) = \nabla f_t(\tilde{\pi}^*), \qquad \nabla_{\varpi} f_{2,t}^{\pi}(\alpha,\tilde{\pi}^*,0) = 0,$$

so that

$$\nabla_{\tilde{\pi}} l_t^{\pi}(\alpha,\tilde{\pi}^*,0) = \frac{\nabla f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)}, \qquad \nabla_{\varpi} l_t^{\pi}(\alpha,\tilde{\pi}^*,0) = 0.$$

Second-order partial derivatives With straightforward differentation we obtain

$$\begin{aligned}
\nabla^2_{\tilde{\pi}\tilde{\pi}'} l_t^{\pi}(\alpha,\tilde{\pi},\varpi) &= \frac{\nabla^2_{\tilde{\pi}\tilde{\pi}'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} - \frac{\nabla_{\tilde{\pi}} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} \frac{\nabla_{\tilde{\pi}'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} \\
\nabla^2_{\tilde{\pi}\varpi'} l_t^{\pi}(\alpha,\tilde{\pi},\varpi) &= \frac{\nabla^2_{\tilde{\pi}\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} - \frac{\nabla_{\tilde{\pi}} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} \frac{\nabla_{\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} \\
\nabla^2_{\varpi\varpi'} l_t^{\pi}(\alpha,\tilde{\pi},\varpi) &= \frac{\nabla^2_{\varpi\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} - \frac{\nabla_{\varpi} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)} \frac{\nabla_{\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}{f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi)}
\end{aligned}$$

with

$$\begin{aligned}
\nabla^2_{\tilde{\pi}\tilde{\pi}'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi) &= \nabla^2_{\tilde{\pi}\tilde{\pi}'}\alpha_{1,t} f_t(\tilde{\phi}) + \nabla_{\tilde{\pi}}\alpha_{1,t}\nabla' f_t(\tilde{\phi}) \\
&\quad + \nabla f_t(\tilde{\phi})\nabla_{\tilde{\pi}'}\alpha_{1,t} + \alpha_{1,t}\nabla^2 f_t(\tilde{\phi}) \\
&\quad - \nabla^2_{\tilde{\pi}\tilde{\pi}'}\alpha_{1,t} f_t(\tilde{\varphi}) - \nabla_{\tilde{\pi}}\alpha_{1,t}\nabla' f_t(\tilde{\varphi}) \\
&\quad - \nabla f_t(\tilde{\varphi})\nabla_{\tilde{\pi}'}\alpha_{1,t} + (1-\alpha_{1,t})\nabla^2 f_t(\tilde{\varphi}) \\
\nabla^2_{\tilde{\pi}\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi) &= \nabla^2_{\tilde{\pi}\varpi'}\alpha_{1,t} f_t(\tilde{\phi}) + \nabla_{\tilde{\pi}}\alpha_{1,t}\nabla' f_t(\tilde{\phi}) D^{(1)}_{\tilde{\phi},\varpi} \\
&\quad + \nabla f_t(\tilde{\phi})\nabla_{\varpi'}\alpha_{1,t} + \alpha_{1,t}\nabla^2 f_t(\tilde{\phi}) D^{(1)}_{\tilde{\phi},\varpi} \\
&\quad - \nabla^2_{\tilde{\pi}\varpi'}\alpha_{1,t} f_t(\tilde{\varphi}) - \nabla_{\tilde{\pi}}\alpha_{1,t}\nabla' f_t(\tilde{\varphi}) D^{(1)}_{\tilde{\varphi},\varpi} \\
&\quad - \nabla f_t(\tilde{\varphi})\nabla_{\varpi'}\alpha_{1,t} + (1-\alpha_{1,t})\nabla^2 f_t(\tilde{\varphi}) D^{(1)}_{\tilde{\varphi},\varpi} \\
\nabla^2_{\varpi\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi},\varpi) &= \nabla^2_{\varpi\varpi'}\alpha_{1,t} f_t(\tilde{\phi}) + \nabla_{\varpi}\alpha_{1,t}\nabla' f_t(\tilde{\phi}) D^{(1)}_{\tilde{\phi},\varpi} \\
&\quad + \alpha_{1,t} D^{(1)\prime}_{\tilde{\phi},\varpi}\nabla^2 f_t(\tilde{\phi}) D^{(1)}_{\tilde{\phi},\varpi} + D^{(1)\prime}_{\tilde{\phi},\varpi}\nabla f_t(\tilde{\phi})\nabla_{\varpi'}\alpha_{1,t} \\
&\quad - \nabla^2_{\varpi\varpi'}\alpha_{1,t} f_t(\tilde{\varphi}) - \nabla_{\varpi}\alpha_{1,t}\nabla' f_t(\tilde{\varphi}) D^{(1)}_{\tilde{\varphi},\varpi} \\
&\quad + (1-\alpha_{1,t}) D^{(1)\prime}_{\tilde{\varphi},\varpi}\nabla^2 f_t(\tilde{\varphi}) D^{(1)}_{\tilde{\varphi},\varpi} - D^{(1)\prime}_{\tilde{\varphi},\varpi}\nabla f_t(\tilde{\varphi})\nabla_{\varpi'}\alpha_{1,t}
\end{aligned}$$

For brevity, we omit the expressions of $\nabla^2_{\tilde{\pi}\tilde{\pi}'}\alpha_{1,t}$, $\nabla^2_{\tilde{\pi}\varpi'}\alpha_{1,t}$, and $\nabla^2_{\varpi\varpi'}\alpha_{1,t}$. Evaluated at $(\alpha,\tilde{\pi},\varpi) = (\alpha,\tilde{\pi}^*,0)$ we get

$$\begin{aligned}
\nabla^2_{\tilde{\pi}\tilde{\pi}'} f_{2,t}^{\pi}(\alpha,\tilde{\pi}^*,0) &= \nabla^2 f_t(\tilde{\pi}^*) \\
\nabla^2_{\tilde{\pi}\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi}^*,0) &= 0 \\
\nabla^2_{\varpi\varpi'} f_{2,t}^{\pi}(\alpha,\tilde{\pi}^*,0) &= \nabla_{\varpi}\alpha^*_{1,t}\nabla'_{(2,\dots,p+2)} f_t(\tilde{\pi}^*) + \nabla_{(2,\dots,p+2)} f_t(\tilde{\pi}^*)\nabla_{\varpi'}\alpha^*_{1,t} + \alpha_1\alpha_2\nabla^2_{(2,\dots,p+2)(2,\dots,p+2)} f_t(\tilde{\pi}^*) \\
&= \nabla_{\varpi}\alpha^*_{1,t}\nabla'_{(2,\dots,p+2)} f_t(\tilde{\pi}^*) + \nabla_{(2,\dots,p+2)} f_t(\tilde{\pi}^*)\nabla_{\varpi'}\alpha^*_{1,t} + \alpha_1\alpha_2\nabla^2_{(2,\dots,p+2)(2,\dots,p+2)} f_t(\tilde{\pi}^*)
\end{aligned}$$

so that

$$\nabla^2_{\tilde{\pi}\tilde{\pi}'} l_t^{\pi}(\alpha, \tilde{\pi}^*, 0) = \frac{\nabla^2 f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)} - \frac{\nabla f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)} \frac{\nabla' f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)}$$

$$\nabla^2_{\tilde{\pi}\varpi'} l_t^{\pi}(\alpha, \tilde{\pi}^*, 0) = 0$$

$$\begin{aligned}\nabla^2_{\varpi\varpi'} l_t^{\pi}(\alpha, \tilde{\pi}^*, 0) &= \frac{\nabla_{\varpi}\alpha^*_{1,t}\nabla'_{(2,\ldots,p+2)} f_t(\tilde{\pi}^*) + \nabla_{(2,\ldots,p+2)} f_t(\tilde{\pi}^*)\nabla_{\varpi'}\alpha^*_{1,t} + \alpha_1\alpha_2\nabla^2_{(2,\ldots,p+2)(2,\ldots,p+2)} f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)} \\ &= \alpha_1\alpha_2\left[\frac{\nabla_{(2,\ldots,p+2)}\mathsf{n}_p(\tilde{\pi}^*)}{\mathsf{n}_p(\tilde{\pi}^*)}\frac{\nabla'_{(2,\ldots,p+2)} f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)} + \frac{\nabla_{(2,\ldots,p+2)} f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)}\frac{\nabla'_{(2,\ldots,p+2)}\mathsf{n}_p(\tilde{\pi}^*)}{\mathsf{n}_p(\tilde{\pi}^*)}\right. \\ &\qquad\left. + \frac{\nabla^2_{(2,\ldots,p+2)(2,\ldots,p+2)} f_t(\tilde{\pi}^*)}{f_t(\tilde{\pi}^*)}\right].\end{aligned}$$

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