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# SUBGEOMETRIC ERGODICITY AND $\beta$-MIXING: SUPPLEMENTARY MATERIAL

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*Proof of Theorem 1.* We use Theorem 2.5(ii) of [2] to prove (a). To this end, first note that, due to the assumed irreducibility and aperiodicity, the petite set $C$ in Condition Drift–G is small ([1], Theorem 5.5.7). We first show that, for some $r > 1$,

$$\sup_{x \in C} \mathbb{E}_x \left[ r^{\tau_C} \right] < \infty;$$

cf. Theorem 2.5(ii) of [2]. We proceed as in the proof of Theorem 15.2.5 in [1] and, for the $\beta$ in Condition Drift–G, choose an $r \in (1, (1-\beta)^{-1})$ and set $\varepsilon = r^{-1} - (1-\beta)$ so that $0 < \varepsilon < \beta$ and $\varepsilon$ is the solution to $r = (1-\beta+\varepsilon)^{-1}$. Now we may reorganize the drift condition as

$$\mathbb{E}\left[V(X_1) \mid X_0 = x\right] \leq r^{-1}V(x) - \varepsilon V(x) + b\mathbf{1}_C(x), \qquad x \in \mathsf{X}.$$

Define $Z_k = r^k V(X_k)$, $k = 0, 1, 2, \ldots$, so that $\mathbb{E}[Z_{k+1} \mid \mathcal{F}_0^k] = r^{k+1}\mathbb{E}[V(X_{k+1}) \mid \mathcal{F}_0^k]$ and thus

$$\mathbb{E}[Z_{k+1} \mid \mathcal{F}_0^k] \leq r^{k+1}\{r^{-1}V(X_k) - \varepsilon V(X_k) + b\mathbf{1}_C(X_k)\} = Z_k - \varepsilon r^{k+1}V(X_k) + r^{k+1}b\mathbf{1}_C(X_k).$$

Applying Proposition 11.3.2 of [1] with $f_k(x) = \varepsilon r^{k+1}V(x)$, $s_k(x) = br^{k+1}\mathbf{1}_C(x)$, and stopping time $\tau_C$ we obtain

$$\mathbb{E}_x\left[\sum_{k=0}^{\tau_C - 1} \varepsilon r^{k+1}V(X_k)\right] \leq V(x) + \mathbb{E}_x\left[\sum_{k=0}^{\tau_C - 1} br^{k+1}\mathbf{1}_C(X_k)\right] = V(x) + br\mathbf{1}_C(x),$$

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because $\mathbf{1}_C(X_1) = \cdots = \mathbf{1}_C(X_{\tau_C - 1}) = 0$ by the definition of $\tau_C$. Multiplying by $\varepsilon^{-1} r^{-1}$ and noting that $V(\cdot) \geq 1$, we obtain, for some finite constants $c_1, c_2$,

$$\mathbb{E}_x \left[ \sum\nolimits_{k=0}^{\tau_C - 1} r^k \right] \leq \mathbb{E}_x \left[ \sum\nolimits_{k=0}^{\tau_C - 1} r^k V(X_k) \right] \leq c_1 V(x) + c_2.$$

As $\sup_{x \in C} V(x) < \infty$, $\sup_{x \in C} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_C - 1} r^k \right] < \infty$ is obtained. Using $\sum_{k=0}^{\tau_C - 1} r^k = (r^{\tau_C} - 1)/(r - 1)$, this is equivalent to $\sup_{x \in C} \mathbb{E}_x [r^{\tau_C}] < \infty$ as desired; note that we also have, for some finite constants $c_3, c_4$, $\mathbb{E}_x [r^{\tau_C}] \leq c_3 V(x) + c_4$. Now Theorem 2.5(ii) of [2] implies that, for some $r_1 > 1$, $\lim_{n \to \infty} r_1^n \| P^n(x\,; \cdot) - \pi(\cdot) \| = 0$, so that the geometric ergodicity of part (a) is established.

To prove (b), suppose the initial state $X_0$ has distribution $\mu$ such that $\int_{\mathsf{X}} \mu(dx) V(x) < \infty$. By Theorem 2.5(iii) of [2] it suffices to prove that $\mathbb{E}_\mu [r^{\tau_C}] < \infty$. As $\mathbb{E}_\mu [r^{\tau_C}] = \int_{\mathsf{X}} \mu(dx) \mathbb{E}_x [r^{\tau_C}]$, the inequality $\mathbb{E}_x [r^{\tau_C}] \leq c_3 V(x) + c_4$ obtained above implies $\mathbb{E}_\mu [r^{\tau_C}] < \infty$ and hence the validity of (b) for some $r_2 > 1$ (Theorem 2.5(iii) of [2]).

Next consider part (d). In the stationary case ($\mu = \pi$), the geometric ergodicity established in (a) and Theorem 2.1 of [2] imply that $\lim_{n \to \infty} \tilde{r}_2^n \int \pi(dx) \| P^n(x\,; \cdot) - \pi(\cdot) \| = 0$ for some $\tilde{r}_2 > 1$ (and condition $\int_{\mathsf{X}} \pi(dx) V(x) < \infty$ is not needed). Thus (b) holds in the stationary case. Regarding part (c) in the stationary case, note from (5) that now $\beta(n) = \int \pi(dx) \| P^n(x\,; \cdot) - \pi(\cdot) \|$, $n = 1, 2, \ldots$, so that (b) and (c) are clearly equivalent (and hold with the same rate $\tilde{r}_2$).

To prove (c) in the general case, recall that $n_1 = \lfloor n/2 \rfloor$ so that $n/2 - 1 < n_1 \leq n/2$, and note that for any $\rho > 1$ and $n \geq 2$, $1 = \rho^{1-n/2} \rho^{n/2-1} < \rho^{1-n/2} \rho^{n_1} = \rho(\rho^{1/2})^{-n} \rho^{n_1}$. Now choose $r_3$ such that $1 < r_3 < \min\{r_2^{1/2}, \tilde{r}_2^{1/2}\}$ (where $r_2$ and $\tilde{r}_2$ are as above in the proofs of parts (b) and (d)). Now use these remarks and the inequality in Lemma A.1 (in the main paper) to obtain

$$r_3^n \beta(n) \leq \frac{1}{2} \tilde{r}_2 (r_3 \tilde{r}_2^{-1/2})^n \tilde{r}_2^{n_1} \int \pi(dx) \| P^{n_1}(x\,; \cdot) - \pi \| + \frac{3}{2} r_2 (r_3 r_2^{-1/2})^n r_2^{n_1} \int \mu(dx) \| P^{n_1}(x\,; \cdot) - \pi \| .$$

From the proofs of (b) and (d) we obtain the results $\lim_{n \to \infty} r_2^{n_1} \int \mu(dx) \| P^{n_1}(x\,; \cdot) - \pi \| = 0$ and $\lim_{n \to \infty} \tilde{r}_2^{n_1} \int \pi(dx) \| P^{n_1}(x\,; \cdot) - \pi \| = 0$, so that $\lim_{n \to \infty} r_3^n \beta(n) = 0$ and hence (c) follows. □

## References

[1] Meyn, S. P. and Tweedie, R. L. (2009). *Markov Chains and Stochastic Stability* 2nd ed. Cambridge University Press, Cambridge.

[2] Nummelin, E. and Tuominen, P. (1982). Geometric ergodicity of Harris recurrent Markov chains with applications to renewal theory. *Stoch. Proc. Appl.* **12,** 187–202.